ArXiv manuscript No.
(will be inserted by the editor)

Ruled Laguerre minimal surfaces

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Received: date / Accepted: date

Abstract A *Laguerre minimal surface* is an immersed surface in $\mathbb{R}^3$ being an extremal of the functional $\int (H^2/K - 1)dA$. In the present paper, we prove that the only *ruled* Laguerre minimal surfaces are up to isometry the surfaces $\mathbf{R}(\varphi, \lambda) = (A\varphi,\, B\varphi,\, C\varphi + D\cos 2\varphi\,) + \lambda\,(\sin\varphi,\, \cos\varphi,\, 0\,)$, where $A, B, C, D \in \mathbb{R}$ are fixed. To achieve invariance under Laguerre transformations, we also derive all Laguerre minimal surfaces that are enveloped by a family of cones. The methodology is based on the isotropic model of Laguerre geometry. In this model a Laguerre minimal surface enveloped by a family of cones corresponds to a graph of a biharmonic function carrying a family of isotropic circles. We classify such functions by showing that the top view of the family of circles is a pencil.

Keywords Laguerre geometry · Laguerre minimal surface · ruled surface · biharmonic function

Mathematics Subject Classification (2000) 53A40 · 49Q10 · 31A30

Contents

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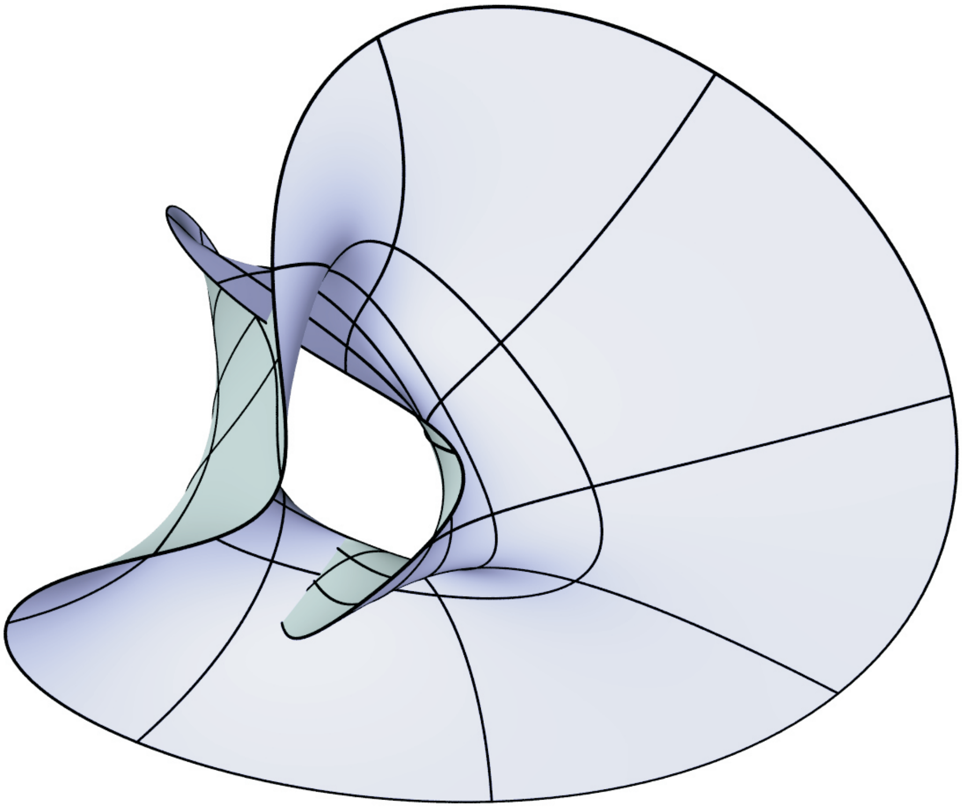

Fig. 1 A general L-minimal surface enveloped by a hyperbolic family of cones. For details refer to Definition 3 and Corollary 5.

1 Introduction

This is the third in a series of papers [19,18] where we develop and study a novel approach to the Laguerre differential geometry of immersed Legendre surfaces in $\mathbb{R}^3$. Laguerre geometry is the Euclidean geometry of oriented planes and spheres. Besides Möbius and Lie geometry, it is one member of the three classical sphere geometries in $\mathbb{R}^3$ [8].

After the seminal work [4] of Blaschke on this topic in the beginning of the 20th century, this classical topic has again found the interest of differential geometers.

For instance, the celebrated work on discrete differential geometry by Bobenko and coworkers [7,6,5] heavily uses this theory in developing discrete counterparts to continuous definitions.

On the practical side, recent research in architectural geometry identified certain classes of polyhedral surfaces, namely conical meshes [13,23] and meshes with edge offsets [21], as particularly suitable for the representation and fabrication of architectural freeform structures. These types of polyhedral surfaces are actually objects of Laguerre sphere geometry [6,13,21,20,28,18].

The aim is to study (discrete, see [18], and continuous, see [19]) minimizers of geometric energies which are invariant under Laguerre transformations. The simplest energy of this type has been introduced by Blaschke [2,3,4]. Using mean curvature H, Gaussian curvature K, and the surface area element dA of an immersion $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$, it can be expressed as the surface integral

$$\Omega = \int (H^2 - K)/K \, dA. \tag{1.1}$$

Though the quantities H, K, A used for the definition are not objects of Laguerre geometry, the functional Ω invariant under Laguerre transformations. An immersion $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ with $K \neq 0$, which is an extremal of the energy Ω with respect to compactly supported variations, is called a *Laguerre-minimal* (*L-minimal*) *surface*. A Euclidean minimal surface (besides a plane) is a particular case of such a surface.

In 1842 Catalan proved that the only ruled Euclidean minimal surfaces are the plane and the helicoid. A surface is *ruled*, if each point of the surface belongs to a line segment contained in the surface. One of the main purposes of this paper is to describe all ruled Laguerre minimal surfaces.

The property of a surface to be ruled is not invariant under Laguerre transformations. A line in a surface may be taken to a cone or cylinder of revolution touching the image of the surface along a curve. Hence, we will also derive all Laguerre minimal surfaces which are *enveloped by a family of cones*; see Figure 1. In the following, when speaking of a cone, we will always assume this to be a cone of revolution, including the special cases of a rotational cylinder and a line.

Our approach is based on a recent result [19] which shows that Laguerre minimal surfaces appear as graphs of biharmonic functions in the isotropic model of Laguerre geometry. This result has various corollaries on Laguerre minimal surfaces, geometric optics and linear elasticity.

A Laguerre minimal surface enveloped by a family of cones corresponds to a graph of a biharmonic function carrying a family of isotropic circles. We classify such functions. In particular we show that besides a few exceptions the top view of such a family of isotropic circles must be a pencil. In the course of the proof of this result we also develop a new symmetry principle for biharmonic functions.

1.1 Previous work

Differential geometry in the three classical sphere geometries of Möbius, Laguerre and Lie, respectively, is the subject of Blaschke's third volume on differential geometry [4]. For a more modern treatment we refer to Cecil [8]. Here we focus on contributions to L-minimal surfaces. Many L-minimal surfaces are found in the work of Blaschke [2,3,4] and in papers by his student König [10,11].

Recently, this topic found again the interest of differential geometers. The stability of L-minimal surfaces has been analyzed by Palmer [16]; he also showed that these surfaces are indeed local minimizers of (1.1). Li and Wang studied L-minimal surfaces using the Laguerre Gauss map and Weierstrass representation [12,29]. Musso and Nicolodi studied L-minimal surfaces by the method of moving frames [15]. L-minimal surfaces which are envelopes of a family of cones include as special cases the L-minimal canal surfaces described by Musso and Nicolodi [14].

1.2 Contributions

Our main result is a description of all the L-minimal surfaces which are envelopes of an analytic family $\mathcal{F}$ of cones of revolution. We show that for any such surface (besides a sphere and a parabolic cyclide) the family $\mathcal{F}$ belongs to one of three simple types; see Definitions 2, 3, 4 and Corollary 2. For each type we represent the surface as a convolution of certain basic surfaces; see Examples 1–11 and Corollaries 4, 5, 6.

As an application we show the following:

Theorem 1 *A ruled Laguerre minimal surface is up to isometry a piece of the surface*

$$\mathbf{R}(\varphi,\lambda) = (A\varphi,\, B\varphi,\, C\varphi + D\cos 2\varphi\,) + \lambda\,(\sin\varphi,\, \cos\varphi,\, 0\,)\,, \tag{1.2}$$

for some $A, B, C, D \in \mathbb{R}$ such that $C^2 + D^2 \neq 0$.

In other words, a ruled L-minimal surface can be constructed as a superposition of a frequency 1 rotating motion of a line in a plane, a frequency 2 "harmonic oscillation" between two lines parallel to the plane, and a constant-speed translation. Equivalently, a ruled L-minimal surface is a convolution of a helicoid, a cycloid, and the Plücker conoid; see Examples 1–3 and Theorem 6 for accurate statement.

One more result of the paper is a description of all the i-Willmore surfaces carrying an analytic family of i-circles; see Table 1 for definitions, and Corollary 1, Theorems 5, 8, 10 for the statements.

1.3 Organization of the paper

In §2 we give an introduction to isotropic and Laguerre geometries and translate the investigated problem to the language of isotropic geometry. This section does not contain new results. In §3 we state and prove the Pencil Theorem 4, which describes the possible families $\mathcal{F}$ of cones. In §4 we describe the Laguerre minimal surfaces for each type of cone family $\mathcal{F}$ and prove Theorem 1.

2 Isotropic model of Laguerre geometry

2.1 Isotropic geometry

Isotropic geometry has been systematically developed by Strubecker [25,26,27] in the 1940s; a good overview of the many results is provided in the monograph by Sachs [24].

The *isotropic space* is the affine space $\mathbb{R}^3$ equipped with the norm $\|(x,y,z)\|_i := \sqrt{x^2+y^2}$. The invariants of affine transformations preserving this norm are subject of *isotropic geometry.*

The projection $(x,y,z) \mapsto (x,y,0)$ of isotropic space onto the xy-plane is called *top view.* Basic objects of isotropic geometry and their definitions (from the point of view of Euclidean geometry in isotropic space) are given in the first two columns of Table 1; see also Figure 2. We return to the third column of the table further.

Table 1 Basic objects of isotropic geometry as images of surfaces in the isotropic model of Laguerre geometry.

Object of isotropic geometry	Definition	Corresponding surface in Laguerre geometry
point	point in isotropic space	oriented plane
non-isotropic line	line nonparallel to the z-axis	cone
non-isotropic plane	plane nonparallel to the z-axis	oriented sphere
i-circle of elliptic type	ellipse whose top view is a circle	cone
i-circle of parabolic type	parabola with z-parallel axis	cone
i-sphere of parabolic type	paraboloid of revolution with z-parallel axis	oriented sphere
i-paraboloid	graph of a quadratic function $z = F(x,y)$	parabolic cyclide *or* oriented sphere
i-Willmore surface	graph of a (multi-valued) biharmonic function $z = F(x,y)$	L-minimal surface

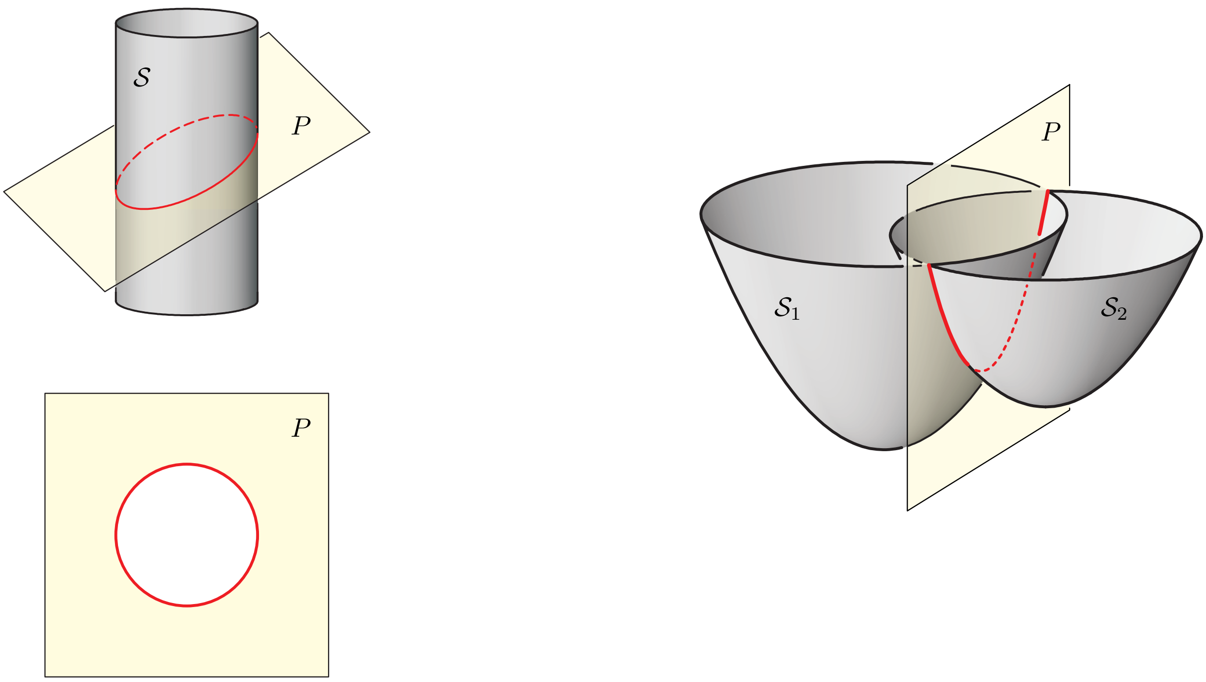

Fig. 2 (Left) An i-circle of elliptic type is the intersection curve of a vertical round cylinder $\mathcal{S}$ and a non-isotropic plane P. When viewed from the top, the i-circle is a Euclidean circle. (Right) An i-circle of parabolic type is a parabola with z-parallel axis. This curve appears as the intersection curve of two i-spheres, $\mathcal{S}_1$ and $\mathcal{S}_2$, of parabolic type with the same i-mean curvature. For more details refer to Table 1 and §2.1.

In isotropic space there exists a counterpart to Möbius geometry. One puts i-spheres of parabolic type and non-isotropic planes into the same class of *isotropic Möbius spheres* (*i-M-spheres*); they are given by the equation $z = \frac{a}{2}(x^2 + y^2) + bx + cy + d$ for some $a, b, c, d \in \mathbb{R}$. The coefficient a in this representation is called the *i-mean curvature* of the i-M-sphere. An intersection curve of two i-M-spheres is called an *i-M-circle*; it may be an i-circle of elliptic or parabolic type or a non-isotropic straight line.

Similarly to Euclidean Möbius geometry, where an ideal point is added to $\mathbb{R}^3$, in isotropic Möbius geometry an *ideal line* ℓ_∞ is added to $\mathbb{R}^3$. The resulting space $\mathbb{R}^3 \cup \ell_\infty$ is called *extended isotropic space*. By definition, an i-M-sphere with i-mean curvature a intersects the ideal line at the point $a \in \ell_\infty$.

A map acting on $\mathbb{R}^3 \cup \ell_\infty$ is called an *isotropic Möbius* (*i-M*) *transformation*, if it takes i-M-spheres to i-M-spheres (and hence i-M-circles to i-M-circles). The top view of an i-M-transformation is a planar Euclidean Möbius transformation. Basic i-M-transformations which together with the translation $(x, y, z) \mapsto (x + 1, y, z)$ generate the whole group of i-M-transformations are given in the first column of Table 2. Here R^θ is the counterclockwise rotation through an angle θ around the z-axis.

Table 2 Basic isotropic Möbius transformations as images of Laguerre transformations in the isotropic model of Laguerre geometry.

i-M-transformation	Corresponding L-transformation
$(x, y, z) \mapsto R^\theta(x, y, z)$	rotation R^θ
$(x, y, z) \mapsto (x, y, z + ax + by)$	translation by vector $(a, b, 0)$
$(x, y, z) \mapsto (x, y, z + x^2 + y^2 - 1)$	translation by vector $(0, 0, 1)$
$(x, y, z) \mapsto (x, y, z + h)$	h-offset operation
$(x, y, z) \mapsto (x, y, az)$	homothety with coefficient a
$(x, y, z) \mapsto (x, y, z)/(x^2 + y^2)$	reflection with respect to the plane $z = 0$
$(x, y, z) \mapsto (x, y, z)/\sqrt{2}$	transformation Λ

2.2 Laguerre Geometry

A *contact element* is a pair (r, P), where r is a point in $\mathbb{R}^3$, and P is an oriented plane passing through the point r. Denote by $ST\mathbb{R}^3$ the space of all contact elements.

To an immersed oriented surface Φ in $\mathbb{R}^3$ assign the set of all the contact elements $(r, P) \in ST\mathbb{R}^3$ such that $r \in \Phi$ and P is the oriented tangent plane to Φ at the point r. We get a *Legendre surface*, i. e., the image of an immersion $(\mathbf{r}, \mathbf{P}) : \mathbb{R}^2 \to ST\mathbb{R}^3$ such that $d\mathbf{r}(u, v) \parallel \mathbf{P}(u, v)$. Further we do not distinguish between an immersed surface in $\mathbb{R}^3$ and the corresponding Legendre surface, if no confusion arises.

An example of a Legendre surface not obtained from an immersed one is a *point*, or a *sphere of radius* 0, which is the set of all the contact elements (r, P) such that $r = r_0$ is fixed and $P \ni r_0$ is arbitrary.

A *Laguerre transformation* (*L-transformation*) is a bijective map $ST\mathbb{R}^3 \to ST\mathbb{R}^3$ taking oriented planes to oriented planes and oriented spheres (possibly of radius 0) to oriented spheres (possibly of radius 0). The invariants of Laguerre transformations are the subject of *Laguerre geometry* [4,8].

Note that an L-transformation does not in general preserve points, since those are seen as spheres of radius 0 and may be mapped to other spheres. A simple example of an L-transformation is the *h-offset operation*, translating a contact element (r, P) by the vector $h\mathbf{n}$, where $\mathbf{n}$ is the positive unit normal vector to the oriented plane P.

A Laguerre transformation is uniquely defined by its action on the set of oriented planes. A *Hesse normal form* of an oriented plane P is the equation $n_1x + n_2y + n_3z + h = 0$ of the plane such that (n_1, n_2, n_3) is the positive unit normal vector to the oriented plane.

Consider the Laguerre transformation Λ taking an oriented plane in the Hesse normal form $n_1x + n_2y + n_3z + h = 0$ to the oriented plane $n_1x + n_2y + \frac{1}{2}(3n_3 + 1)z + h = 0$ with obvious orientation. Denote by $\tilde{\mathbf{r}}(u, v)$ the surface obtained from a surface $\mathbf{r}(u, v)$ by the transformation and parametrized so that the tangent planes to the surfaces $\tilde{\mathbf{r}}(u, v)$ and $\mathbf{r}(u, v)$ are parallel at points having the same parameters u and v. This notation is convenient in our classification results which follow.

Similarly, denote by $\mathbf{r}^\theta(u, v)$ the surface obtained from a surface $\mathbf{r}(u, v)$ by the rotation R^θ and parametrized so that the tangent planes to the surfaces $\mathbf{r}^\theta(u, v)$ and $\mathbf{r}(u, v)$ are parallel for each $(u, v) \in \mathbb{R}^2$.

More examples of Laguerre transformations are given in the second column of Table 2.

For a pair of parallel oriented planes P_1 and P_2 in Hesse normal forms $n_1x + n_2y + n_3z + h_1 = 0$ and $n_1x + n_2y + n_3z + h_2 = 0$, denote by $a_1P_1 \oplus a_2P_2$ the plane in Hesse normal form $n_1x + n_2y + n_3z + a_1h_1 + a_2h_2 = 0$. Define a *convolution surface* $a_1\Phi_1 \oplus a_2\Phi_2$ of two Legendre surfaces Φ_1 and Φ_2 to be the Legendre surface formed by the contact elements of the form $(a_1r_1 + a_2r_2, a_1P_1 \oplus a_2P_2)$, where $(r_1, P_1) \in \Phi_1$ and $(r_2, P_2) \in \Phi_2$ run through all the pairs of contact elements with P_1 parallel to P_2.

2.3 Isotropic model of Laguerre geometry

To each oriented plane in the Hesse normal form $n_1x+n_2y+n_3z+h = 0$, with $n_3 \neq -1$ assign the point

$$\frac{1}{n_3+1}(n_1, n_2, h) \tag{2.1}$$

of the isotropic space. To an oriented plane in the Hesse normal form $-z + h = 0$ assign the ideal point $h \in \ell_\infty$. This induces a map from the space $ST\mathbb{R}^3$ to the extended isotropic space $\mathbb{R}^3 \cup \ell_\infty$. The map provides the *isotropic model of Laguerre geometry*. For a more geometric definition see [22].

A nondevelopable surface Φ viewed as set of oriented tangent planes is mapped to a surface Φ^i in the isotropic model. Conversely, the surface Φ can be reconstructed given the surface Φ^i:

Proposition 1 cf. [19, Corollary 2] *Let Φ be a nondevelopable immersed surface. Suppose that Φ^i is a graph of multi-valued function $z = F(x, y)$. Then the surface Φ can be parametrized as follows:*

$$\mathbf{r}(x,y) = \frac{1}{x^2+y^2+1}\begin{pmatrix} (x^2-y^2-1)F_x + 2xyF_y - 2xF \\ (y^2-x^2-1)F_y + 2xyF_x - 2yF \\ 2xF_x + 2yF_y - 2F \end{pmatrix}. \tag{2.2}$$

As examples, consider the pairs of surfaces Φ^i and Φ given in the last two columns of Table 1.

An oriented sphere with center (m_1, m_2, m_3) and radius R is mapped to the isotropic Möbius sphere

$$z = \frac{R+m_3}{2}(x^2+y^2) - m_1x - m_2y + \frac{R-m_3}{2}. \tag{2.3}$$

A cone viewed as the common tangent planes of two oriented spheres is mapped to the common points of two i-M-spheres (= i-M-circle) in the isotropic model. In particular, a line is mapped to an i-M-circle of the form

$$\begin{cases} z = m_3(x^2+y^2-1) - m_1x - m_2y, \\ z = n_3(x^2+y^2-1) - n_1x - n_2y. \end{cases} \tag{2.4}$$

This leads to the following key observation; see Figure 3:

Proposition 2 *Let Φ be a nondevelopable immersed surface. The surface Φ is enveloped by a family of cones if and only if the surface Φ^i is the union of a family of i-M-circles.*

Theorem 2 [19, Theorem 1] *Let Φ be a nondevelopable immersed surface. The surface Φ is Laguerre minimal if and only if the surface Φ^i is a graph of a multi-valued biharmonic function $z = F(x, y)$, i. e., a function satisfying the equation $\Delta(\Delta(F)) = 0$.*

Convolution surface $a_1\Phi_1 \oplus a_2\Phi_2$ corresponds in the isotropic model to the linear combination of the two multi-valued functions whose graphs are Φ_1^i and Φ_2^i. Thus a convolution surface of two L-minimal surfaces is L-minimal [19, Corollary 3].

L-transformations correspond to i-M-transformations in the isotropic model and vice versa. Some examples are given in Table 2. Invariance of L-minimal surfaces under L-transformations is translated in the isotropic model as follows:

Theorem 3 [19, Theorem 1] *Suppose that F is a graph of a function biharmonic in a region $U \subset \mathbb{R}^2$ and $m : \mathbb{R}^3 \cup \ell_\infty \to \mathbb{R}^3 \cup \ell_\infty$ is an isotropic Möbius transformation. Then $m(F)$ is a graph of a function biharmonic in the top view of $m(U \times \mathbb{R}) - \ell_\infty$.*

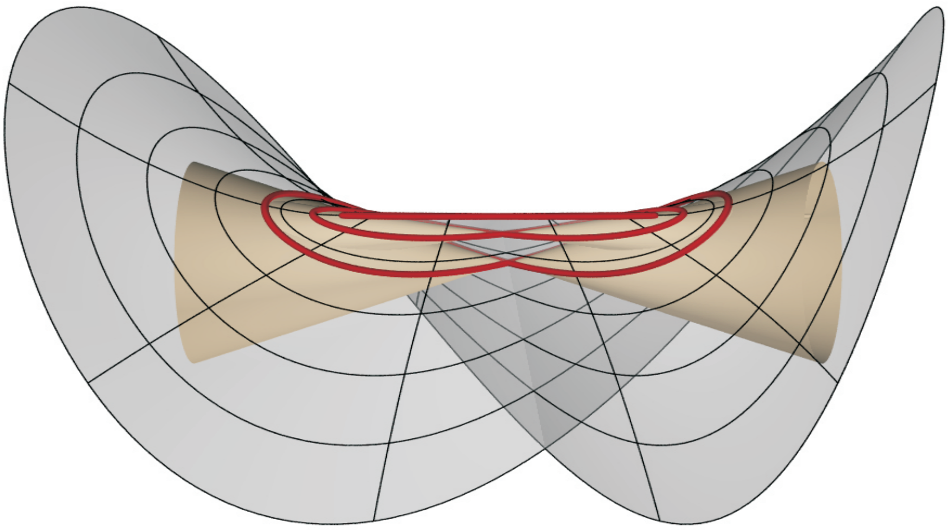

Fig. 3 The L-minimal surface $\mathbf{r}_5$ arising as the envelope of a family of cones; see Example 5 for the details.

Plan of the proof of Theorem 1 To a ruled L-minimal surface there corresponds an i-Willmore surface containing a family of i-M-circles in the isotropic model.

First we show that the top view of the family of i-M-circles is a pencil. In other words, all the rulings of the L-minimal surface are parallel to one plane.

Then by appropriate choice of coordinates we transform the pencil into a pencil of lines. In the latter case we describe all possible i-Willmore surfaces by solving the biharmonic equation explicitly.

Returning to the Euclidean model we get a description of all ruled L-minimal surfaces.

3 Biharmonic functions carrying a family of i-circles

3.1 Statement of the Pencil theorem

In this section we show that the top view of a family of i-M circles contained in a graph of a biharmonic function (besides a few exceptions) is a *pencil*, i. e., a set of circles orthogonal to two fixed ones. This implies that the rulings of a ruled L-minimal surface are parallel to one plane. Denote by $I = [0; 1]$.

Theorem 4 (Pencil theorem) *Let $F(x, y)$ be a biharmonic function in a region $U \subset \mathbb{R}^2$. Let S_t, $t \in I$, be an analytic family of circles in the plane. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then either S_t, $t \in I$, is a pencil of circles or*

$$F(x,y) = A((x-a)^2 + (y-b)^2) + \frac{B(x-c)^2 + C(x-c)(y-d) + D(y-d)^2}{(x-c)^2 + (y-d)^2} \tag{3.1}$$

for some $a, b, c, d, A, B, C, D \in \mathbb{R}$.

The exceptional function (3.1) has the following property: there is a 2-parametric family of circles S_t, $t \in I^2$, such that for each $t \in I^2$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function.

Plan of the proof of Pencil Theorem 4. We say that two circles *cross* each other if their intersection consists of 2 points. Assume that the family of circles is not a pencil. Then it contains a subfamily of one of the following types:

(1) the circles S_t, $t \in I$, pairwise cross but do not pass through one point;
(2) the circles S_t, $t \in I$, have a common point O;
(3) the circles S_t, $t \in I$, are nested.

First we prove the theorem in case when the region U is sufficiently large, i. e., $U \supset \bigcup S_t$, $U \supset \bigcup S_t - \{O\}$ and $U = \mathbb{R}^2$ for types (1), (2) and (3), respectively.

Then we reduce the theorem to the latter case by a biharmonic continuation of the function F; see Figure 4. The continuation is in 2 steps.

In the first step we extend the function F *along* the circles S_t until we reach the envelope of the family of circles (if the envelope is nonempty). This is done easily for arbitrary real analytic function F. The main difficulty is that extending F along the circles beyond the envelope may lead to a multi-valued function.

In the second step we extend the function F *across* the circles S_t to make the region U sufficiently large keeping the function single-valued. This is done using a new symmetry principle for biharmonic functions.

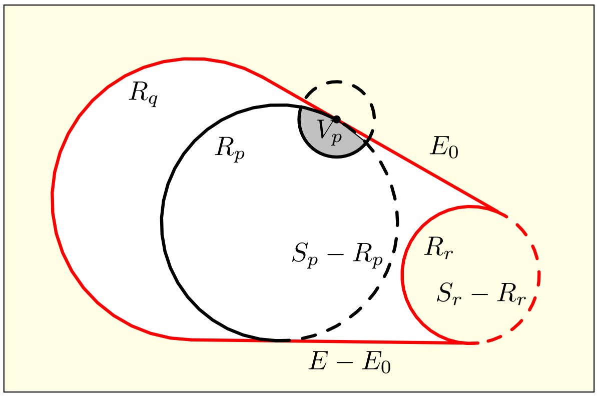

Fig. 4 Biharmonic continuation of a function F whose restriction to an arc of each circle S_t, $t \in I$, is linear. First we extend F along the circles to the white region bounded by certain arcs $R_q \subset S_q$, $R_r \subset S_r$ and two pieces of the envelope E. Then we extend F across the circles to a neighborhood of the piece E_0 of the envelope, using reflection of the gray region V_p with respect a circle S_p. For details refer to Lemma 7 and its proof.

3.2 Three typical cases

First let us prove Theorem 4 in three typical particular cases treated in Lemmas 1, 2 and 3 for "sufficiently large" sets U.

Lemma 1 (Crossing circles) *Let S_t, $t \in I$, be a family of pairwise crossing circles in the plane distinct from a pencil of circles. Let F be an arbitrary function defined in the set $U = \bigcup_{t\in I} S_t$. Suppose that for each $t \in I$ the restriction $F|_{S_t}$ is a restriction of a linear function. Then $F = A((x-a)^2 + (y-b)^2) + B$ for some $a, b, A, B \in \mathbb{R}$.*

Proof Denote by l_t the linear function $F|_{S_t}$. Let $s_t = 0$ be the normalized equation of the circle S_t, i. e., $s_t = x^2 + y^2 + \dots$ and $s_1|_{S_1} = 0$. For any pair $s, t \in I$ both differences $s_t - s_s$ and $l_t - l_s$ are linear functions vanishing on $S_s \cap S_t$. Thus $l_t - l_s = k_{st}(s_t - s_s)$ for some number k_{st}.

Since the family S_t is not a pencil it follows that there are 3 circles S_1, S_2, S_3 in the family such that the functions s_1, s_2, s_3 are linearly independent. Let us show that $F = k_{12}s_1 + l_1$.

Indeed, in the circle S_1 we have $F = l_1 = k_{12}s_1 + l_1$ because $s_1|_{S_1} = 0$. In the circle S_2 we have $F = l_2 = k_{12}s_2 + l_2 = k_{12}s_1 + l_1$ by definition of the number k_{12}.

Consider the circle S_3. We have $k_{23} = k_{31} = k_{12}$ because otherwise $k_{12}(s_1 - s_2) + k_{23}(s_2 - s_3) + k_{31}(s_3 - s_1) = (l_1 - l_2) + (l_2 - l_3) + (l_3 - l_1) = 0$ is a nontrivial linear combination of s_1, s_2, s_3. Thus in the circle S_3 we have $F = l_3 = k_{13}s_3 + l_3 = k_{13}s_1 + l_1 = k_{12}s_1 + l_1$.

Finally, take any circle S_t. We can replace one of the functions s_1, s_2, s_3 by s_t to get still a linearly independent triple. Repeating the argument from the previous paragraph we get $F = k_{12}s_1 + l_1$ in S_t. Thus $F = k_{12}s_1 + l_1$ in the whole set U. $\square$

Lemma 2 (Circles with a common point) *Let S_t, $t \in I$, be a family of pairwise crossing circles in the plane passing through the origin O. Assume that no three circles of the family belong to one pencil. Let F be an arbitrary function defined in the set $U = \bigcup_{t\in I} S_t - \{O\}$. Suppose that for each $t \in I$ the restriction $F\,|_{S_t-\{O\}}$ is a restriction of a linear function. Then*

$$F(x, y) = A((x-a)^2 + (y-b)^2) + \frac{Bx^2 + Cxy + Dy^2}{x^2 + y^2}$$

for some $a, b, A, B, C, D \in \mathbb{R}$.

Proof Perform the transformation $(x, y, z) \mapsto (x, y, z)/(x^2 + y^2)$. Then the family of circles S_t transforms to a family of lines L_t. By the assumptions of the lemma any two of the lines L_t intersect each other but no three of the lines L_t pass through one point. The graph of the function F transforms to a graph of a function G defined in $V = \bigcup_{t\in I} L_t$. For each $t \in I$ the restriction $G\,|_{L_t}$ is a quadratic function.

Take three lines L_1, L_2, L_3 from the family. Let l_1, l_2, l_3 be nonzero linear functions vanishing in the lines L_1, L_2, L_3, respectively. Let l be a linear function such that $l = F$ in the points $L_1 \cap L_2, L_2 \cap L_3, L_3 \cap L_1$. Since $G\,|_{L_1}$ is quadratic and $G - l = 0$ in the points $L_1 \cap L_2$ and $L_1 \cap L_3$ it follows that $G\,|_{L_1} = k_{23} l_2 l_3 + l$ for some number k_{23}. Analogously, $G\,|_{L_2} = k_{31} l_3 l_1 + l$ and $G\,|_{L_3} = k_{12} l_1 l_2 + l$ for some numbers k_{12} and k_{31}.

Let us prove that $G = k_{12} l_1 l_2 + k_{23} l_2 l_3 + k_{31} l_3 l_1 + l$ in the whole set V. Indeed, consider the difference $H = k_{12} l_1 l_2 + k_{23} l_2 l_3 + k_{31} l_3 l_1 + l - G$. Then $H\,|_{L_1} = 0$, $H\,|_{L_2} = 0$, $H\,|_{L_3} = 0$ by the above. Take a line L_t distinct from L_1, L_2, L_3. Then $H\,|_{L_t}$ is a quadratic function. On the other hand, $H(L_t \cap L_1) = H(L_t \cap L_2) = H(L_t \cap L_3) = 0$. Since the points $L_t \cap L_1, L_t \cap L_2, L_t \cap L_3$ are pairwise distinct it follows that $H\,|_{L_t} = 0$. So the function H vanishes in each line L_t. Thus $H = 0$ in the set V.

We have proved that G is a polynomial of degree not greater than 2. Performing the inverse transformation $(x, y, z) \mapsto (x, y, z)/(x^2 + y^2)$ we obtain the required formula for the function F. □

Lemma 3 (Nested circles) *Let S_1 and S_2 be the pair of circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Let F be a function biharmonic in the whole plane $\mathbb{R}^2$. Suppose that for each $t = 1, 2$ the restriction $F\,|_{S_t}$ is a restriction of a linear function. Then*

$$F(x, y) = (x^2 + y^2)(Ax + By + C) + ax + by + c$$

for some $a, b, c, A, B, C \in \mathbb{R}$.

The function $F(x, y) = (x^2 + y^2)\log(x^2 + y^2)$ extended by $F(0, 0) = 0$ might seem to be a counterexample to this lemma but in fact it is not: $\partial^2 F/\partial x^2$ is discontinuous at the origin.

Proof By Proposition 5 below it follows that there are functions u_1, u_2, harmonic in $\mathbb{R}^2$, such that $F = (x^2 + y^2 - 2)u_1 + u_2$. Then $F = (x^2 + y^2 - 2)u_3 + (x^2 + y^2 - 1)u_2$, where the function $u_3 = u_1 - u_2$ is also harmonic in $\mathbb{R}^2$. Since u_2 is harmonic in the whole plane $\mathbb{R}^2$ and the restriction $u_2\,|_{S_2} = F\,|_{S_2}$ is linear it follows by uniqueness theorem that the function u_2 itself is linear. Analogously u_3 is linear and the lemma follows. □

Proposition 3 *Let $F(x, y) = (x^2 + y^2)(Ax + By + C) + ax + by + c$, where $A^2 + B^2 \neq 0$. Suppose that the restriction of the function F to a circle $S \subset \mathbb{R}^2$ is linear. Then the center of the circle S is the origin.*

Proof It suffices to consider the case when $a = b = c = C = 0$. Let $x^2 + y^2 + px + qy + r = 0$ be the equation of S. If the restriction $F\,|_S$ is linear then S is a projection of (a part of) the intersection of the surface $z = F(x, y)$ and a plane $z = \alpha x + \beta y + \gamma$. Thus there exist numbers k, l, m such that

$$(x^2 + y^2)(Ax + By) - (\alpha x + \beta y + \gamma) = (kx + ly + m)(x^2 + y^2 + px + qy + r).$$

We get

$$k = A, \quad l = B, \quad pk + m = 0, \quad ql + m = 0, \quad pl + qk = 0.$$

Thus $p = -An$, $q = Bn$, $m = A^2 n = -B^2 n$ for some $n \in \mathbb{R}$. Since $A^2 + B^2 \neq 0$ it follows that $n = 0$. So S is the circle $x^2 + y^2 + r = 0$. □

3.3 Biharmonic continuation

We are going to reduce Theorem 4 to Lemmas 1, 2 and 3 by "biharmonic continuation" of the function F. We say that a function F biharmonic in a region U *extends* to a function G biharmonic in a region V if there is an open set $D \subset U \cap V$ such that $F = G$ in D. Notice that F can be distinct from G in $U \cap V$ if the latter set is disconnected.

Proposition 4 (Uniqueness of a continuation) *If two functions biharmonic in a region $V \subset \mathbb{R}^2$ coincide in a region $U \subset V$ then these functions coincide in the region V.*

Proof Let F, G be functions such that $\Delta^2 F = \Delta^2 G = 0$ in V and $F = G$ in U. Then $\Delta(F - G)$ is harmonic in V and vanishes in U. Thus $\Delta(F - G) = 0$ in V. Hence $F - G$ is harmonic in V and vanishes in U. Thus $F - G = 0$ in V. □

Lemma 4 (Continuation along circles) *Let $F : U \to \mathbb{R}$ be a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Let S_t, $t \in I$, be an analytic family of circles in the plane containing at least two distinct ones. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then for some segment $J = [q, r] \subset I$ the function F extends to a function biharmonic in a region bounded by certain arcs of the circles S_q, S_r and possibly two pieces of the envelope of the family S_t, $t \in J$.*

Proof The idea of the proof is to extend the function linearly along the circles until we reach the envelope. The obtained function will be a real analytic continuation of the initial function and hence it will be biharmonic.

Let E be the envelope of the family S_t, $t \in I$. For fixed $t \in I$ and a small $\epsilon > 0$ a *local piece* of the envelope E is the envelope of the family S_τ, $\tau \in [t - \epsilon, t + \epsilon]$. The envelope or its local piece can be empty.

Each circle S_t touches (each local piece of) the envelope E at most at 2 points, depending on the arrangement of the circles sufficiently close to S_t. Let $R_t \subset S_t$ be one of the open arcs joining the touching points of the circle S_t and (each local piece of) the envelope E. Let R_t be one of the sets $S_t - O_t$ or $\emptyset$ (respectively, $R_t = S_t$ or $\emptyset$), if there is a unique such touching point O_t (respectively, no such touching points). Choose the arcs R_t so that they form a continuous family.

One can assume that for each t in a segment $J_1 \subset I$ we have $R_t \cap U \neq \emptyset$. Indeed, if $R_t \cap U \neq \emptyset$ for at least one $t \in I$ then the same condition holds in a neighborhood J_1 of t. Otherwise replace each R_t by $S_t - R_t$ and repeat the argument.

There is a segment $[q, r] = J \subset J_1$ such that the arcs R_t, $t \in J$, are pairwise disjoint. Indeed, if $E = \emptyset$ then one can take $J = J_1$. If a local piece of the envelope E is a pair of smooth curves then one can approximate the family S_t, $t \in I$, by a family of circles touching a pair of lines and get the required segment J. If a piece of the envelope degenerates to a point then one can find a segment J analogously.

So $V = \bigcup_{t \in J} R_t$ is a region bounded by arcs R_q, R_r and possibly two pieces of the envelope E.

By the assumption of the lemma the restriction $F|_{R_t \cap U}$ is the restriction of a linear function for each $t \in J$. Extend the function F linearly to each arc R_t. We get a function defined in the whole region V. It remains to prove that the obtained function is biharmonic in V.

Let us show that F is real analytic in V. Parametrize the arc R_t by the functions $x(t, \phi) = x_0(t) + r(t) \cos \phi$, $y(t, \phi) = y_0(t) + r(t) \sin \phi$. Consider (t, ϕ) as coordinates in V. Since the family S_t is analytic it follows that these coordinates are analytic. Without loss of generality assume $[q, r] \times [\alpha, \beta] \subset U$ for some $\alpha, \beta \in [-\pi, \pi]$. Then $F(t, \phi)$ is real analytic in $[q, r] \times [\alpha, \beta]$. By the construction $F(t, \phi) = a(t) \cos \phi + b(t) \sin \phi + c(t)$ in the region V for some functions $a(t), b(t), c(t)$. Thus $a(t), b(t), c(t)$ are real analytic in $[q, r]$. Hence F is real analytic in the whole region V.

Then the function $\Delta^2 F$ is also real analytic in the region V and vanishes in the open set $U \cap V$. By the uniqueness theorem for analytic functions it follows that $\Delta^2 F = 0$ in the whole region V, i. e., F is biharmonic in V. □

To extend the function F further we need the following preparations.

Proposition 5 (Representation) [1] *Let $s(x, y) = x^2 + y^2 + ax + by + c$. Then any function F biharmonic in a simply-connected region $U \subset \mathbb{R}^2$ can be represented as $F = su_1 + u_2$ for some functions u_1, u_2 harmonic in U.*

Proposition 6 (Arc extension) *Let $S \subset R$ be a pair of circular arcs. Let F be a biharmonic function defined in a neighborhood of the arc R. Suppose that $F|_S$ is a restriction of a linear function. Then $F|_R$ is the restriction of the same linear function.*

Proof Let l be the linear function $F|_S$. Let $s(x,y) = 0$ be the normalized equation of the circle containing the arc S. By Proposition 5 we have $F - l = su_1 + u_2$ for some functions u_1, u_2 harmonic in a neighborhood U of the arc R. Then $u_2|_S = (F - l)|_S = 0$. By the symmetry principle for harmonic functions it follows that $u_2(x,y) = -u_2(x',y')$ for any pair of points $(x,y), (x',y') \in U$ symmetric with respect to the circle $s(x,y) = 0$. In particular, $u_2|_R = 0$. Thus $F|_R = l$. □

Now we are going to give a version of a symmetry principle for biharmonic functions. The usual symmetry principle [17,9] is not applicable in our situation because we have no information on the growth of the function in the normal directions to the circles.

We use the following notation. Let S_s and S_t be a pair of circles. Denote by O_t the center of the circle S_t and by $r_t : \mathbb{R}^2 - \{O_t\} \to \mathbb{R}^2 - \{O_t\}$ the reflection with respect to the circle S_t. Let $r_t(U)$ be a shorthand for $r_t(U - \{O_t\})$. Denote by Σ_{st} the *limit set* of the pencil of circles passing through S_s and S_t, i. e., $\Sigma_{st} = \{\, x \in \mathbb{R}^2 : r_s(x) = r_t(x) \,\}$. If $S_s \neq S_t$ then the limit set consists of at most 2 points.

The following technical definition is required to keep the function single-valued during the continuation process with our symmetry principle.

Definition 1 (Nicely arranged region) A region $U \subset \mathbb{R}^2$ is *nicely arranged* with respect to two circles S_1 and S_2, if the set $U \cap r_1(U) \cap r_2(U) - \Sigma_{12}$ has a connected component D such that $S_1 \cap D \neq \emptyset$ and $S_2 \cap D \neq \emptyset$.

Notice that an arbitrary region is nicely arranged with respect to any pair of sufficiently close circles intersecting the region.

Lemma 5 (Double symmetry principle) *Let F be a function biharmonic in a simply-connected region $U \subset \mathbb{R}^2$ nicely arranged with respect to a pair of circles $S_1 \neq S_2$. Suppose that for each $t = 1, 2$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then F extends to a function biharmonic in the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$.*

Proof Let l_t be the linear function $F|_{S_t}$ for $t = 1, 2$. Without loss of generality assume that S_1 is the unit circle $x^2 + y^2 = 1$. By Proposition 5 it follows that $F = (x^2 + y^2 - 1)u_2 + u_1 + l_1$ for some functions u_1 and u_2 harmonic in U.

Take functions $\nu_1(z)$ and $\nu_2(z)$ complex analytic in U such that $u_t = \nu_t(z) + \overline{\nu_t(z)}$ for $t = 1, 2$. Since U is simply-connected it follows that $\nu_1(z)$ and $\nu_2(z)$ are single-valued. Let $\lambda_1(z)$ and $\lambda_2(z)$ be linear functions such that $l_t = \lambda_t(z) + \overline{\lambda_t(z)}$ for $t = 1, 2$. For $t = 1, 2$ represent the reflection with respect to the circle S_t as a map $z \mapsto \rho_t(\bar{z})$ for a fractional linear function $\rho_t(z)$.

Let us extend the function u_1 to the open set $r_1(U)$. (What we do is the usual symmetry principle.) Let D be the open set from Definition 1. For each $z \in S_1$ we have $z = \rho_1(\bar{z})$. Thus the condition $F|_{S_1} = l_1$ is equivalent to

$$\nu_1(z) = -\overline{\nu_1(\rho_1(\bar{z}))} \tag{3.2}$$

for each $z \in S_1 \cap D$. Both sides of formula (3.2) are complex analytic functions in D. By the uniqueness theorem it follows that these functions coincide in D. Thus formula (3.2) defines an extension of the function $\nu_1(z)$ to the open set $r_1(U)$. So $u_1 = \nu_1(z) + \overline{\nu_1(z)}$ is the required extension of the function u_1.

Let us extend the function u_2 to the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$. For each $z \in S_2$ we have $z = \rho_2(\bar{z})$. For each $z \in D$ formula (3.2) holds by the previous paragraph. Thus for each $z \in S_2 \cap D$ the condition $F|_{S_2} = l_2$ is equivalent to the condition

$$\nu_2(z) = -\overline{\nu_2(\rho_2(\bar{z}))} + \frac{\overline{\nu_1(\rho_1(\bar{z}))} - \overline{\nu_1(\rho_2(\bar{z}))} - \lambda_1(z) - \overline{\lambda_1(\rho_2(\bar{z}))} + \lambda_2(z) + \overline{\lambda_2(\rho_2(\bar{z}))}}{z\overline{\rho_2(\bar{z})} - 1}. \tag{3.3}$$

Since both sides of formula (3.3) are complex analytic functions in D it follows that these functions coincide in D. If $z \in r_1(U) \cap r_2(U)$ then $\rho_1(\bar{z}), \rho_2(\bar{z}) \in U$. Thus the right-hand side of formula (3.3) defines a function complex analytic in $r_1(U) \cap r_2(U) - \Sigma_{12}$ because the denominator may vanish only in Σ_{12}. Extend the function $\nu_2(z)$ to the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$ by formula (3.3). Then $u_2(z) = \nu_2(z) + \overline{\nu_2(z)}$ is the required extension of the function u_2.

Since both functions u_1 and u_2 extend to $r_1(U) \cap r_2(U) - \Sigma_{12}$ it follows that $F = (x^2 + y^2 - 1)u_2 + u_1 + l_1$ also extends to $r_1(U) \cap r_2(U) - \Sigma_{12}$. □

Lemma 6 (Continuation across nested circles) *Let S_t, $t \in I$, be a family of nested circles in the plane distinct from a pencil of circles. Let $F : U \to \mathbb{R}$ be a function biharmonic in the ring U between S_0 and S_1. Suppose that for each $t \in I$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then the function F extends to a function biharmonic in the whole plane $\mathbb{R}^2$.*

Proof The idea of the proof is to extend the function, using the double symmetry principle, to the ring between $r_0(S_1)$ and S_1, then to the ring between $r_0(S_1)$ and $r_1 r_0(S_1)$, and so on.

Take any pair of circles S_t and S_s, where $s, t \in I$ are sufficiently close to 0. Draw disjoint slits T and T' such that the regions $U - T$ and $U - T'$ are simply-connected. By Lemma 5 the function F extends to both $r_t(U - T) \cap r_s(U - T)$ and $r_t(U - T') \cap r_s(U - T')$. Thus it extends to a (possibly multi-valued) function biharmonic in the ring $r_t(U) \cap r_s(U) \cup U$. The latter function is single-valued because a continuation along the closed path S_0 leads to the initial value. Approaching $t, s \to 0$ one can extend the function F to the ring between the circles $r_0(S_1)$ and S_1. Now approaching $t, s \to 1$ one can extend the function F to the larger ring between the circles $r_0(S_1)$ and $r_1 r_0(S_1)$. Continuing this process one extends F to a function biharmonic in $\mathbb{R}^2$ except the limit set Σ_{01} of the pencil of circles passing through S_0 and S_1.

Since S_t, $t \in I$, is not a pencil of circles it follows that $\Sigma_{0p} \cap \Sigma_{01} = \emptyset$ for some $p \in I$. Repeating the above reflection process for the pair of circles S_0 and S_p one extends the function F to a function biharmonic in the whole plane $\mathbb{R}^2$. □

Lemma 7 (Continuation across crossing circles) *Let S_t, $t \in I$, be an analytic family of pairwise crossing circles in the plane distinct from a pencil. Let $F : U \to \mathbb{R}$ be a function biharmonic in a region $U \subset \mathbb{R}^2$. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F|_{S_t}$ is a restriction of a linear function. Then for some segment $J \subset I$ the function F extends to a function biharmonic in a neighborhood of $\bigcup_{t \in J} S_t$ possibly except a common point of all the circles S_t.*

Proof The idea of the proof is to extend the function first along the circles until we reach the envelope E of the family, then by the double symmetry principle — to a neighborhood of the envelope, and finally — along the circles beyond the envelope; see Figure 4.

By Lemma 4 it follows that F extends to a region bounded by certain arcs of the circles S_q and S_r and two pieces of the envelope E for some $q, r \in I$. Since the family S_t, $t \in [q, r]$, is not a pencil it follows that at least one component E_0 of the envelope E does not degenerate to a point.

Let us extend the function F to a neighborhood of the curve E_0. Use the notation from the proof of Lemma 4. Take $p \in [q, r]$ such that the curve E_0 is smooth at the point $R_p \cap E_0$. (Then in a neighborhood of the the point $R_p \cap E_0$ the family R_t, $t \in [p - \epsilon, p + \epsilon]$, is isotopic to a family of arcs tangent to one line.) Let V_p be the intersection of the open disc bounded by the circle S_p and an open disc of centered at $R_p \cap E_0$. Clearly, if the latter disc is sufficiently small then $V_p \subset U$. Without loss of generality assume that $R_t \cap V_p = \emptyset$ for each $t \in [p - \epsilon, p]$ and $R_t \cap V_p \neq \emptyset$ for each $t \in [p, p + \epsilon]$, where $\epsilon > 0$ is small enough.

Take a pair of circles S_t and S_s, where $s, t \in [p, p + \epsilon]$. By Lemma 5 the function F extends to the region $r_t(V_p) \cap r_s(V_p) - \Sigma_{st}$. Approaching $t, s \to p$ one extends the function F to the region $r_p(V_p)$, and hence to a neighborhood of the point $R_p \cap E_0$. So F extends to a neighborhood V of the curve E_0.

A consequence of this extension is that $F|_{S_t \cap V}$ is linear for each $t \in I$, because each intersection $S_t \cap V$ is connected and Proposition 6 can be applied.

Let us extend the function F along the arcs $S_t - R_t$. Choose $U' \subset U$ and $I' \subset I$ so that $R_t \cap U' = \emptyset$ and $S_t - R_t \cap U' \neq \emptyset$ for each $t \in I'$. Applying Lemma 4 to the region U' and the family S_t, $t \in I'$, we extend the function F to a region bounded by $S_{q'} - R_{q'}$, $S_{r'} - R_{r'}$ and certain pieces of the envelope E for some $p', q' \in I'$. Take a segment J strictly inside $[q', r']$. Then the function F extends to a (possibly multi-valued) function biharmonic in a neighborhood of $\bigcup_{t \in J} S_t$ possibly except a common point of all the circles S_t, $t \in J$. The latter function is single-valued because a continuation along any closed path S_t, $t \in J$, leads to the initial value. □

3.4 Proof and corollaries of the Pencil Theorem

Proof (of Theorem 4) Assume that S_t, $t \in I$, is not a pencil of circles. Clearly, there is a segment $J \subset I$ such that one of the following conditions hold:

(1) the circles S_t, $t \in J$, pairwise cross but do not pass through one point;
(2) the circles S_t, $t \in J$, have a common point O but no three circles S_t, $t \in J$, belong to one pencil;

(3) the circles S_t, $t \in J$, are nested.

Consider each case separately.

Case (1). By Lemma 7 the function F extends to a function biharmonic in a neighborhood of the set $\bigcup_{t\in J_1} S_t$ for some segment $J_1 \subset J$. By Proposition 6 for each $t \in J_1$ the restriction $F|_{S_t}$ is linear. Then by Lemma 1 case (1) follows.

Case (2). Analogously to the previous paragraph case (2) follows from Lemmas 7, 2 and Proposition 6.

Case (3). By Lemma 4 it follows that the function F extends to the ring between a pair of circles from the family. By Theorem 3 we may assume that these two circles are $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Then by Lemma 6 the function F extends to the whole plane $\mathbb{R}^2$. Thus case (3) follows from Lemma 3 and Proposition 3. □

The following corollaries of Theorem 4 are straightforward; see Table 1.

Corollary 1 *Let Φ^i be an i-Willmore surface carrying an analytic family $\mathcal{F}^i$ of i-M-circles. Then either the surface Φ^i is i-M-equivalent to an i-paraboloid or the top view of the family $\mathcal{F}^i$ is a pencil of circles or lines.*

Corollary 2 *Let Φ be an L-minimal surface enveloped by an analytic family $\mathcal{F}$ of cones. Then either the surface Φ is a parabolic cyclide or a sphere, or the Gaussian spherical image of the family $\mathcal{F}$ is a pencil of circles in the unit sphere.*

Corollary 3 *A ruled L-minimal surface is a Catalan surface, i. e., contains a family of line segments parallel to one plane.*

Proof (of Corollary 3) Let Φ be a ruled L-minimal surface. It suffices to prove that the family of lines contained in Φ is analytic; then by Corollary 2 the result follows. Since Φ is L-minimal by Theorem 2 and Proposition 1 it follows that Φ itself is analytic. Since Φ is ruled it follows that for each point $r \in \Phi$ there is a line $L_r \subset \Phi$. Since the direction of the line L_r is an asymptotic direction of the surface Φ at the point r it follows that the family of lines contained in Φ is analytic. □

Remark 1 Theorem 4 does not remain true for biharmonic functions $\mathbb{C}^2 \to \mathbb{C}$. For instance, for the function $F(x, y) = (x^2 + y^2)(x + iy)$ there is a 2-parametric family of circles S_t, $t \in I^2$, such that for each $t \in I^2$ the restriction $F|_{S_t}$ is a restriction of a linear function.

Remark 2 Theorem 4 does not remain true for real analytic functions $\mathbb{R}^2 \to \mathbb{R}$. For instance, the restriction of the function $F(x, y) = \sqrt{(x^2 + y^2)^2 - x^2 + 1}$ to each circle of the family $x^2 + y^2 - tx - \sqrt{t^2 - 1} = 0$ is a restriction of a linear function.

Remark 3 The proof of Theorem 4 is simpler in the generic case when the biharmonic function F extends to a (possibly multi-valued) function in the whole plane except a discrete subset Σ. For instance, to prove Lemma 7 in this case it suffices to take a segment $J \subset I$ such that $S_t \cap \Sigma = \emptyset$ for each $t \in J$.

4 Classification of L-minimal surfaces enveloped by a family of cones

4.1 Elliptic families of cones

The results of the previous section give enough information to describe all the L-minimal surfaces enveloped by a family of cones, in particular, ruled L-minimal surfaces. We have got to know that either the top view of a family of i-M-circles in an i-Willmore is a pencil or the surface contains another family of i-circles with top view being a pencil. Let us consider separately each possible type of the pencil.

Definition 2 An *elliptic* pencil of circles in the plane (or in a sphere) is the set of all the circles passing through two fixed distinct points. A 1-parametric family of cones (possibly degenerating to cylinders or lines) in space is *elliptic* if the Gaussian spherical images of the cones form an elliptic pencil of circles in the unit sphere.

Denote by $\operatorname{Arctan} x = \{\arctan x + \pi k \,:\, k \in \mathbb{Z}\}$ the multi-valued inverse of the tangent function.

Theorem 5 *Let Φ^i be an i-Willmore surface carrying a family of i-M-circles. Suppose that the top view of the family is an elliptic pencil of circles. Then the surface Φ^i is i-M-equivalent to a piece of the surface*

$$z = \big(a_1(x^2+y^2) + a_2 x + a_3\big) \operatorname{Arctan} \frac{y}{x} + \frac{b_1 y^2 + b_2 xy}{x^2+y^2} + c_1 y^2 + c_2 xy \tag{4.1}$$

for some $a_1, a_2, a_3, b_1, b_2, c_1, c_2 \in \mathbb{R}$.

Proof (of Theorem 5) Perform an i-M-transformation taking the elliptic pencil of circles in the top view to the pencil of lines $y = tx$, where t runs through a segment $J \subset \mathbb{R}$. Denote by $z = F(x, y)$ the surface obtained from the surface Φ^i by the transformation, where F is a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Assume without loss of generality that $(0,0) \notin U$ and F is single-valued in U. Since an i-M-transformation takes i-M-circles to i-M-circles it follows that the restriction of the function F to (an appropriate segment of) each line $y = tx$, where $t \in J$, is a quadratic function.

Proposition 7 *Let $F(x, y)$ be a biharmonic function in a region $U \subset \mathbb{R}^2 - \{(0,0)\}$. Suppose that the restriction of the function F to the intersection of each line $y = tx$, where $t \in J$, with the region U is a quadratic function. Then*

$$\begin{aligned} F(x,y) = \big(a_1(x^2+y^2) + a_2 x + a_3 + a_4 y\big) \arctan \frac{y}{x} + \\ + \frac{b_1 y^2 + b_2 xy + b_3 x^2}{x^2+y^2} + c_1 y^2 + c_2 xy + c_3 x^2 + d_1 x + d_2 y \end{aligned} \tag{4.2}$$

for some $a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2 \in \mathbb{R}$.

Proof Consider the polar coordinates in U. Restrict the function F to a subregion of the form $(r_1, r_2) \times (\phi_1, \phi_2) \subset U$. Then $F(r, \phi) = a(\phi) r^2 + b(\phi) r + c(\phi)$ in the region $(r_1, r_2) \times (\phi_1, \phi_2)$ for some smooth functions $a(\phi), b(\phi), c(\phi)$. Thus $r^4 \Delta^2 F = \big(4a'' + a^{(4)}\big) r^2 + \big(b + 2b'' + b^{(4)}\big) r + \big(4c'' + c^{(4)}\big)$. Since $\Delta^2 F = 0$ it follows that the coefficients of this polynomial in r vanish. Solving the obtained ordinary differential equations we get:

$$\begin{aligned} a(\phi) &= \alpha_1 + \alpha_2 \phi + \alpha_3 \cos 2\phi + \alpha_4 \sin 2\phi; \\ b(\phi) &= \beta_1 \cos \phi + \beta_2 \sin \phi + \beta_3 \phi \cos \phi + \beta_4 \phi \sin \phi; \\ c(\phi) &= \gamma_1 + \gamma_2 \phi + \gamma_3 \cos 2\phi + \gamma_4 \sin 2\phi, \end{aligned}$$

for some $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4, \gamma_1, \dots, \gamma_4 \in \mathbb{R}$. Returning to the initial cartesian coordinate system we get the required formula. □

To complete the proof of Theorem 5 perform an appropriate rotation around the z-axis to achieve $a_4 = 0$ in formula (4.2) and then the i-M-transformation $z \mapsto z - c_3(x^2+y^2) - d_1 x - d_2 y - b_3$ to achieve $b_3 = c_3 = d_1 = d_2 = 0$. □

Table 3 Biharmonic functions whose restrictions to each line $y = tx$, $t \in I$, are quadratic functions and corresponding Laguerre minimal surfaces (or a Legendre surface in case of $\mathbf{r}_2$)

Biharmonic function	Laguerre minimal surface
$(x^2+y^2-1)\operatorname{Arctan}(y/x)$	$\mathbf{r}_1(u,v)$
$(x^2+y^2-2)\operatorname{Arctan}(y/x)/2\sqrt{2}$	$\tilde{\mathbf{r}}_1(u,v)$
$-x\operatorname{Arctan}(y/x)$	$\mathbf{r}_2(u,v)$
$(x\cos\theta + y\sin\theta)^2(1 - 1/(x^2+y^2))/2$	$\mathbf{r}_3^{\theta}(u,v)$
$(x\cos\vartheta + y\sin\vartheta)^2(1 - 2/(x^2+y^2))/4\sqrt{2}$	$\tilde{\mathbf{r}}_3^{\vartheta}(u,v)$
$a(x^2+y^2) + bx + cy + d$	oriented sphere

A Laguerre minimal surface enveloped by an elliptic family of cones is obtained from the surface (4.1) by transformation from Proposition 1. Let us give some typical examples obtained from graphs of the functions in the left column of Table 3; see also Figure 5. These examples are "building blocks" whose

convolutions form all the surfaces in question. We represent them in special parametric form $\mathbf{r}(u,v)$, where the map $\mathbf{r}(u,v)$ is the inverse of the composition of the Gaussian spherical map and the stereographic projection. This is convenient to get easy expressions for the convolution surfaces. The choice of building blocks is a question of taste; we choose them to get the simplest possible expressions for $\mathbf{r}(u,v)$.

Example 1 The first building block is the well-known helicoid which is given implicitly by $x = -y\tan(z/2)$. It can be parametrized via

$$\mathbf{r}_1(u,v) = \left(u - \frac{u}{u^2+v^2},\, \frac{v}{u^2+v^2} - v,\, 2\mathrm{Arctan}\,\frac{u}{v} \right) \qquad \text{or as a ruled surface via} \tag{4.3}$$

$$\mathbf{R}_1(\varphi,\lambda) = (\,0,\,0,\,-2\varphi\,) + \lambda\,(\,\sin\varphi,\,\cos\varphi,\,0\,)\,. \tag{4.4}$$

Example 2 The next example is the cycloid $\mathbf{r}(t) = (t-\sin t, 1-\cos t, 0)/2$. One should think of a *cycloid* as a Legendre surface formed by all the contact elements (r,P) such that the plane P passes through the line tangent to the curve $\mathbf{r}(t)$ at the point r of the curve; see the definitions in §2.2. We use the parametrization:

$$\mathbf{r}_2(u,v) = \left(\mathrm{Arctan}\,\frac{u}{v} - \frac{uv}{u^2+v^2},\, \frac{u^2}{u^2+v^2},\, 0 \right). \tag{4.5}$$

The family of tangent lines to the cycloid can be parametrized via

$$\mathbf{R}_2(\varphi,\lambda) = (\,\varphi,\,1,\,0\,) + \lambda\,(\,\sin\varphi,\,\cos\varphi,\,0\,)\,. \tag{4.6}$$

Example 3 The third building block is the Plücker conoid $z = y^2/(x^2+y^2)$. In parametric form it can be written as:

$$\mathbf{r}_3(u,v) = \frac{uv}{u^2+v^2}\left(\frac{v}{u^2+v^2} - v,\, u - \frac{u}{u^2+v^2},\, \frac{u}{v} \right) \qquad \text{or as} \tag{4.7}$$

$$\mathbf{R}_3(\varphi,\lambda) = (\,0,\,0,\,\cos 2\varphi + 1\,)/2 \;+\; \lambda\,(\,\sin\varphi,\,\cos\varphi,\,0\,)\,. \tag{4.8}$$

The Plücker conoid has the special property that it arises as an L-minimal ruled surface and at the same time as an i-Willmore surface carrying a 2-parametric family of i-M-circles.

We shall now see that an arbitrary ruled L-minimal surface is up to isometry a convolution of these building blocks; see Figure 6 to the bottom. Denote by $\mathbf{r}^\theta(u,v) = R^\theta \mathbf{r}(R^{-\theta}(u,v))$, where R^θ is the counterclockwise rotation through an angle θ around the z-axis; in this formula the plane (u,v) is identified with the plane $z=0$.

Theorem 6 (Classification of ruled L-minimal surfaces) *A ruled Laguerre minimal surface is up to isometry a piece of the surface*

$$\mathbf{r}(u,v) = a_1\mathbf{r}_1(u,v) + a_2\mathbf{r}_2(u,v) + a_3\mathbf{r}_3^\theta(u,v) \tag{4.9}$$

for some $a_1, a_2, a_3, \theta \in \mathbb{R}$ such that $a_1^2 + a_3^2 \neq 0$. Conversely, any immersed piece of the surface (4.9) is ruled and Laguerre minimal.

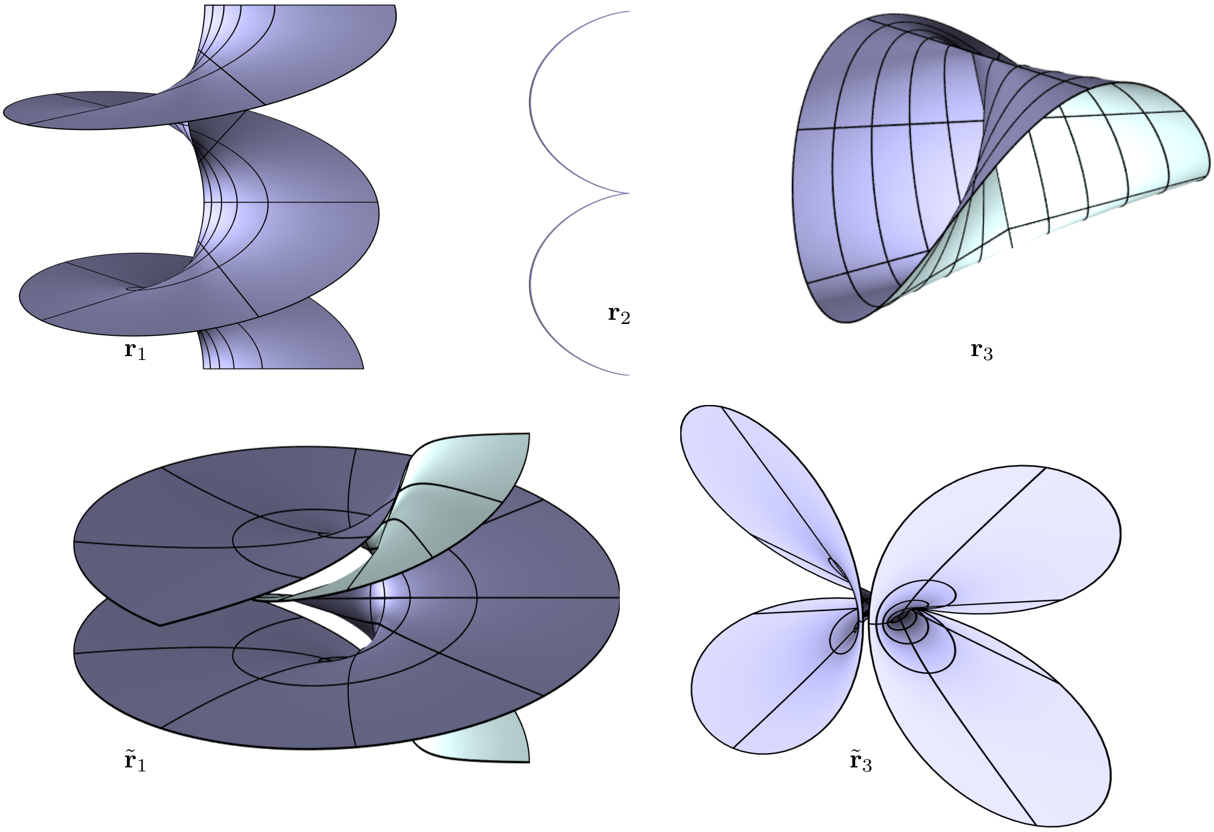

Fig. 5 Building blocks for L-minimal surfaces enveloped by an elliptic family of cones. Starting from the top left we show the surfaces $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\tilde{\mathbf{r}}_3$ and $\tilde{\mathbf{r}}_1$ in clockwise direction. Note that the cycloid $\mathbf{r}_2$ lies in a plane orthogonal to the z-axis. For details refer to Examples 1, 2, 3 and Section 2.2.

Proof (of Theorem 6) Let us prove the direct implication. Let Φ be a ruled L-minimal surface. By Corollary 3 it follows that Φ contains a family of line segments parallel to one plane. Choose a coordinate system so that the plane is Oxy.

Consider the corresponding surface Φ^i in the isotropic model. By Theorem 2 it follows that (a piece of) the surface Φ^i is a graph of a function F biharmonic in a region $U \subset \mathbb{R}^2$. Since the surface Φ carries a family of lines parallel to the plane Oxy it follows that the surface Φ^i carries a family of i-M-circles of the form (2.4) with $m_3 = n_3$. Thus the restriction of the function F to the intersection of the region U with each line $y = tx$, where t runs through a segment $J \subset \mathbb{R}$, is a quadratic function $m_3(t)(x^2 + y^2 - 1) - m_1(t)x - m_2(t)y$.

By Proposition 7 it follows that formula (4.2) holds. In this formula $a_1 + a_3 = b_1 + c_1 = b_2 + c_2 = b_3 + c_3 = 0$ because the restriction of the function F to the lines $y = tx$ has special form $m_3(t)(x^2 + y^2 - 1) - m_1(t)x - m_2(t)y$.

Let us simplify expression (4.2) by appropriate isometries of $\mathbb{R}^3$ (corresponding to i-M-transformations of the isotropic model; see Table 2). First perform an appropriate rotation of $\mathbb{R}^3$ around the z-axis to achieve $a_4 = 0$ in formula (4.2) and appropriate translations along the x- and y-axes to achieve $d_1 = d_2 = 0$. Bringing to the diagonal form one gets $c_1x^2 + c_2xy + c_3y^2 = a\left(x\sin\theta + y\cos\theta\right)^2 + c\left(x^2 + y^2\right)$ for some numbers $a, \theta, c \in \mathbb{R}$. Perform the translation by vector $(0, 0, -c)$ along the z-axis.

After all the above isometries the function (7) becomes a linear combination in the first, third and fourth functions in the left column of Table 3. By Proposition 1 transformation (2.2) takes the functions in the left column of Table 3 to the surfaces in the right column. Since the expression in the right-hand side of the formula (2.2) is linear in F the direct implication in the theorem follows. We exclude the case $a_1 = a_3 = 0$ because it does not lead to an immersion.

Proposition 8 *If $a_1^2 + a_3^2 \neq 0$ then the surface (4.9) contains the family of lines*

$$\mathbf{R}(\varphi, \lambda) = a_1\mathbf{R}_1(\varphi, \lambda) + a_2\mathbf{R}_2(\varphi, \lambda) + a_3 R^\theta \mathbf{R}_3(\varphi - \theta, \lambda), \tag{4.10}$$

where φ is the family parameter and λ is the line parameter.

Proof (of Proposition 8) Fix a number $\varphi \in \mathbb{R}$. Consider three parallel lines $\mathbf{R}_1(\varphi, \lambda)$, $\mathbf{R}_2(\varphi, \lambda)$, and $R^\theta\mathbf{R}_3(\varphi - \theta, \lambda)$. Then the line $\mathbf{R}(\varphi, \lambda)$ given by (4.10) is their convolution as Legendre surfaces (to a line L we assign the Legendre surface $\{\,(r, P) \in ST\mathbb{R}^3 \,:\, r \in L, P \supset L\,\}$). Since the lines $\mathbf{R}_1(\varphi, \lambda)$ and $\mathbf{R}_3(\varphi, \lambda)$ are contained in the surfaces $\mathbf{r}_1(u, v)$ and $\mathbf{r}_3(u, v)$, respectively, and the line $\mathbf{R}_2(\varphi, \lambda)$ is tangent to the curve $\mathbf{r}_2(u, v)$ it follows that the line $\mathbf{R}(\varphi, \lambda)$ is contained in the convolution surface $\mathbf{r}(u, v)$ unless $a_1 = a_3 = 0$. □

Now complete the proof of Theorem 6 by checking its reciprocal implication. By Proposition 8 it follows that the surface (4.9) is ruled unless $a_1 = a_3 = 0$ (when neither piece of the surface is immersed). By Proposition 1, Theorem 2, and Table 3 it follows that any immersed piece of the surface (4.9) is L-minimal. □

Proof (of Theorem 1) By Theorem 6 a ruled L-minimal surface can up to isometry be parametrized via (4.9) with $a_1^2 + a_3^2 \neq 0$. By Proposition 8 the surface can also be parametrized via (4.10). It remains to notice that up to isometry formulas (4.10) and (1.2) define the same class of surfaces. □

An L-minimal surface enveloped by an elliptic family of cones can be obtained from Examples 1–3 by performing L-transformations and taking special convolution surfaces (in general convolution operation does not preserve the class of surfaces enveloped by a family of cones); see Figure 6 to the top. Recall that $\tilde{\mathbf{r}}(u, v)$ is the surface obtained from a surface $\mathbf{r}(u, v)$ by the L-transformation Λ; see §2.2 for the definition.

Corollary 4 (Classification for elliptic type) *A Laguerre minimal surface enveloped by an elliptic family of cones is Laguerre equivalent to a piece of the surface*

$$\mathbf{r}(u, v) = a_1\mathbf{r}_1(u, v) + a_2\mathbf{r}_2(u, v) + a_3\mathbf{r}_3^\theta(u, v) + a_4\tilde{\mathbf{r}}_1(u, v) + a_5\tilde{\mathbf{r}}_3^\vartheta(u, v) \tag{4.11}$$

for some $a_1, a_2, a_3, a_4, a_5, \theta, \vartheta \in \mathbb{R}$. Conversely, an immersed piece of surface (4.11) is Laguerre minimal and is enveloped by an elliptic family of cones.

Proof (of Corollary 4) Let us prove the direct implication. Let Φ be a Laguerre minimal surface enveloped by an elliptic family of cones. Then the surface Φ^i carries a family of i-M-circles such that the top view of the family is an elliptic pencil. By Theorem 5 it follows that the surface Φ^i is i-M-equivalent to surface (4.1).

The right-hand side of formula (4.1) is a linear combination in the expressions in the left column of Table 3. Performing an i-M-transformation $z \mapsto z + a(x^2 + y^2) + bx + cy + d$ one can eliminate the last expression from the linear combination. By Proposition 1 and Table 2 transformation $\Phi^i \mapsto \Phi$ takes the functions in the left column of Table 3 to the surfaces in the right column. Since the expression in the right-hand side of formula (2.2) is linear in F the direct implication follows.

The converse implication follows from Propositions 1, 2, Theorem 2, and Table 3. □

Description of the families of cones. Let describe the families of cones which make up the L-minimal surfaces in question. We view a cone as a linear family of oriented spheres. If we map an oriented sphere with midpoint (m_1, m_2, m_3) and signed radius R to the point $(m_1, m_2, m_3, R) \in \mathbb{R}^4$, we get a correspondence between cones in $\mathbb{R}^3$ and lines in $\mathbb{R}^4$. Surfaces enveloped by a family of cones can be regarded as ruled 2-surfaces in $\mathbb{R}^4$. Laguerre transformations of $\mathbb{R}^3$ correspond to Lorentz transformations of $\mathbb{R}^4$ under this mapping. This is known as the *cyclographic model* of Laguerre geometry; see [22] for more information. We will refer to a ruled 2-surface in $\mathbb{R}^4$ corresponding to a surface enveloped by a family of cones as a *cyclographic preimage*.

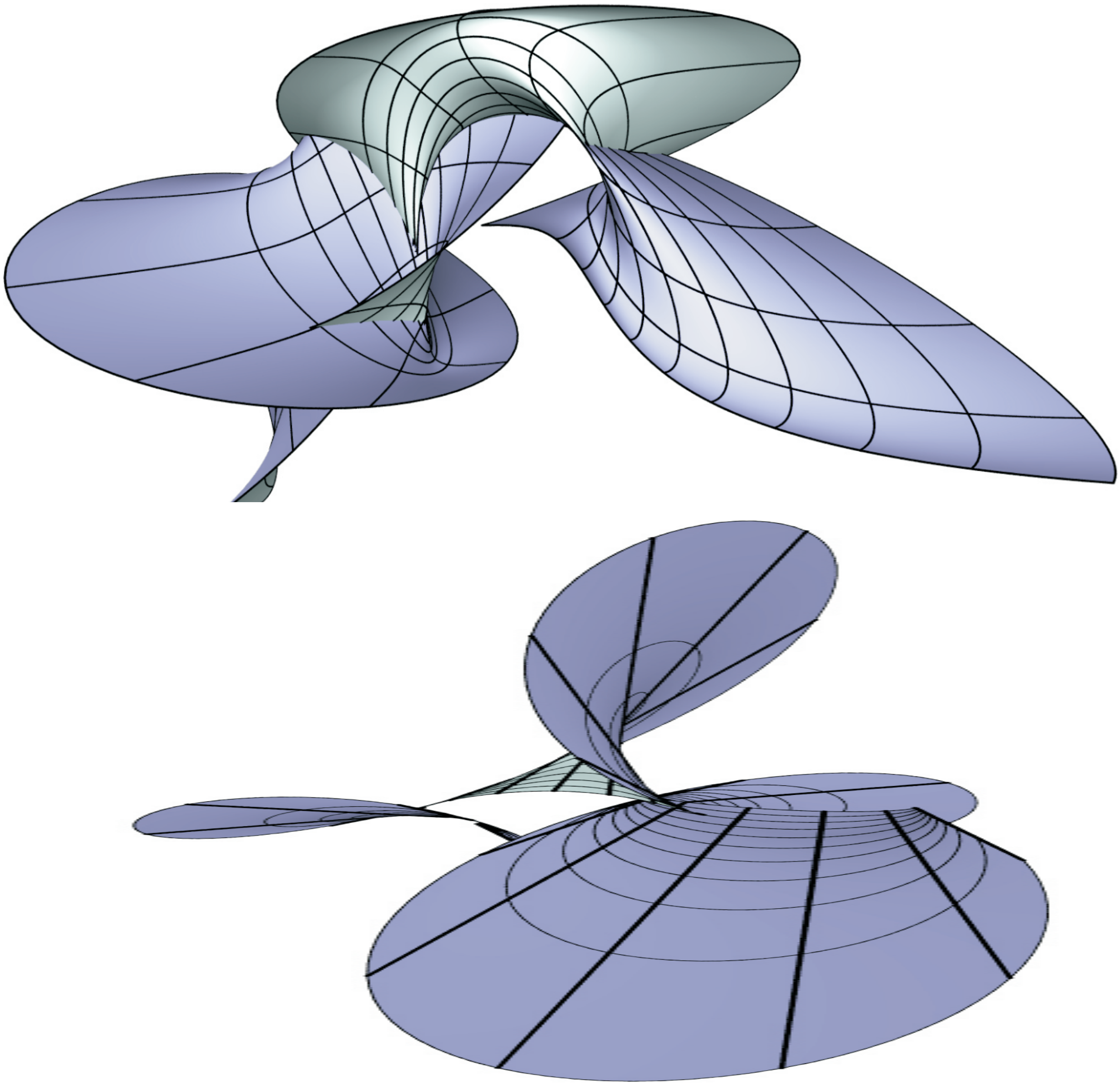

Fig. 6 (Top) A general L-minimal surface enveloped by an elliptic family of cones. For more information refer to Definition 2 and Corollary 4. (Bottom) A general ruled L-minimal surface is a convolution surface of the surfaces $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The rulings are depicted in black. For more information refer to Theorem 6.

Proposition 9 *The cyclographic preimage of the surface* $\tilde{\mathbf{r}}_1(u,v)$ *can be parametrized as*

$$\tilde{\mathbf{R}}_1(\varphi,\lambda) = (0,\,0,\,-3\varphi,\,\varphi)/2\sqrt{2} \,+\, \lambda\,(\sin\varphi,\,\cos\varphi,\,0,\,0)\,. \tag{4.12}$$

The cyclographic preimage of the surface $\tilde{\mathbf{r}}_3(u,v)$ *can be parametrized as*

$$\tilde{\mathbf{R}}_3(\varphi,\lambda) = \left(0,\,0,\,3\cos^2\varphi,-\cos^2\varphi\right)/4\sqrt{2} \,+\, \lambda\,(\sin\varphi,\,\cos\varphi,\,0,\,0)\,. \tag{4.13}$$

Proof (of Proposition 9) Let us find the cyclographic preimage of the surface $\tilde{\mathbf{r}}_1(u,v)$. Consider the image Φ^i of the surface in the isotropic model. By Table 3 the image Φ^i has the equation $z = (x^2+y^2-2)\mathrm{Arctan}(y/x)/2\sqrt{2}$. Thus for each $\varphi \in \mathbb{R}$ the surface Φ^i contains the i-circle

$$\begin{cases} z = -(x^2+y^2-2)\varphi/2\sqrt{2}, \\ 0 = x\sin\varphi + y\cos\varphi. \end{cases}$$

By formula (2.3) this i-circle is the image (in the isotropic model) of the limit of the sequence of the cones touching the two spheres with cyclographic coordinates $(0,\, 0,\, -3\varphi,\, \varphi)\,/2\sqrt{2}$ and $N\,(\sin\varphi,\, \cos\varphi,\, 0,\, 0)$, where $N \to \infty$. Thus the cyclographic preimage of the surface $\tilde{\mathbf{r}}_1(u,v)$ is given by the required formula (4.12). The cyclographic preimages of the surface $\tilde{\mathbf{r}}_3(u,v)$ and all the other surfaces below are computed analogously. □

Theorem 7 *The cyclographic preimage of a Laguerre minimal surface enveloped by an elliptic family of cones is up to Lorentz transformations a piece of the surface*

$$\mathbf{R}(\varphi,\lambda) = (\, A\varphi,\, B\varphi,\, C\varphi + D\cos 2\varphi,\, E\varphi + F\cos 2\varphi + G\sin 2\varphi\,) + \lambda\,(\,\sin\varphi,\, \cos\varphi,\, 0,\, 0\,) \tag{4.14}$$

for some $A, B, C, D, E, F, G \in \mathbb{R}$.

In other words, an L-minimal surface enveloped by an elliptic family of cones can be interpreted as a frequency 1 rotation of a line in a plane, plus a frequency 2 "harmonic oscillation", and a constant-speed translation; this time in $\mathbb{R}^4$.

Proof Notice that the cyclographic preimage of a ruled surface in $\mathbb{R}^3$ is the surface itself, if $\mathbb{R}^3$ is identified with subspace $R = 0$ of $\mathbb{R}^4$. Analogously to the proof of Proposition 8 one can show that the cyclographic preimage of surface (4.11) can be parametrized via

$$\mathbf{R}(\varphi,\lambda) = a_1\mathbf{R}_1(\varphi,\lambda) + a_2\mathbf{R}_2(\varphi,\lambda) + a_3R^{\theta}\mathbf{R}_3(\varphi-\theta,\lambda) + a_4\tilde{\mathbf{R}}_1(\varphi,\lambda) + a_5R^{\vartheta}\tilde{\mathbf{R}}_3(\varphi-\vartheta,\lambda), \tag{4.15}$$

with the same $a_1, a_2, a_3, a_4, a_5, \theta, \vartheta \in \mathbb{R}$. By Examples 1–3 and Proposition 9 such a parametrization gives the same class of surfaces as the required parametrization (4.14). □

4.2 Hyperbolic families of cones

Here we consider the second kind of L-minimal surfaces enveloped by a family of cones.

Definition 3 A *hyperbolic* pencil of circles in the plane (or in a sphere) is the set of all the circles orthogonal to two fixed crossing circles. A 1-parametric family of cones (possibly degenerating to cylinders or lines) in space is *hyperbolic* if the Gaussian spherical images of the cones form a hyperbolic pencil of circles in the unit sphere.

Theorem 8 *Let Φ^i be an i-Willmore surface carrying a family of i-M-circles. Suppose that the top view of the family is a hyperbolic pencil of circles. Then the surface Φ^i is i-M-equivalent to a piece of the surface*

$$z = \big(a_1(x^2+y^2) + a_2x + a_3\big)\ln(x^2+y^2) + \frac{b_1y+b_2x}{x^2+y^2} + (c_1y+c_2x)(x^2+y^2) \tag{4.16}$$

for some $a_1, a_2, a_3, b_1, b_2, c_1, c_2 \in \mathbb{R}$.

Proof (of Theorem 8) Perform an i-M-transformation taking the hyperbolic pencil of circles in the top view into the pencil of concentric circles $x^2+y^2=t$, where t runs through a segment $J \subset \mathbb{R}$. Denote by $z = F(x,y)$ the surface obtained from the surface Φ^i by the transformation, where F is a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Since an i-M-transformation takes i-M-circles to i-M-circles it follows that the restriction of the function F to (an appropriate arc of) each circle $x^2+y^2=t$, where $t \in J$, is a linear function.

Without loss of generality assume that $(0,0) \notin U$. Consider the polar coordinates in U. Then $F(r,\phi) = a(r)\cos\phi + b(r)\sin\phi + c(r)$. Thus

$$\begin{aligned} r^4\Delta^2F = r^4F_{rrrr} + 2r^3F_{rrr} - r^2F_{rr} + rF_r + 2r^2F_{rr\phi\phi} - 2rF_{r\phi\phi} + 4F_{\phi\phi} + F_{\phi\phi\phi\phi} = \\ = \Big(r^4a^{(4)} + 2r^3a^{(3)} - 3r^2a'' + 3ra' - 3a\Big)\cos\phi + \\ + \Big(r^4b^{(4)} + 2r^3b^{(3)} - 3r^2b'' + 3rb' - 3b\Big)\sin\phi + \\ + \Big(r^4c^{(4)} + 2r^3c^{(3)} - r^2c'' + rc'\Big)\,. \end{aligned}$$

Table 4 Biharmonic functions whose restrictions to each circle $x^2 + y^2 = t$, $t \in I$, are linear functions and corresponding Laguerre minimal surfaces

Biharmonic function	Laguerre minimal surface
$(x^2+y^2-1)(\ln(x^2+y^2)-2)/2-2$	$\mathbf{r}_4(u,v)$
$(x^2+y^2-2)(\ln(x^2+y^2)-2-\ln 2)/4\sqrt{2}-\sqrt{2}$	$\tilde{\mathbf{r}}_4(u,v)$
$x\ln(x^2+y^2)+x(x^2+y^2-1)$	$\mathbf{r}_5(u,v)$
$(x\cos\theta+y\sin\theta)(x^2+y^2-2+1/(x^2+y^2))$	$\mathbf{r}_6^\theta(u,v)$
$(x\cos\vartheta+y\sin\vartheta)(x^2+y^2-4+4/(x^2+y^2))/4$	$\tilde{\mathbf{r}}_6^\vartheta(u,v)$
$a(x^2+y^2)+bx+cy+d$	oriented sphere

Since $\Delta^2 F = 0$ it follows that the coefficients of this trigonometric polynomial vanish. Solving the obtained ordinary differential equations we get:

$$a(r) = \alpha_1 r + \alpha_2 r\ln r + \alpha_3/r + \alpha_4 r^3;$$
$$b(r) = \beta_1 r + \beta_2 r\ln r + \beta_3/r + \beta_4 r^3;$$
$$c(r) = \gamma_1 + \gamma_2 r^2 + \gamma_3 \ln r + \gamma_4 r^2 \ln r.$$

One can achieve $\beta_2 = 0$ by an appropriate rotation of the coordinate system around the origin. One can also achieve $\alpha_1 = \beta_1 = \gamma_1 = \gamma_2 = 0$ by the i-M-transformation $z \mapsto z - \gamma_2(x^2+y^2) - \alpha_1 x - \beta_1 y - \gamma_1$ Returning to the cartesian coordinate system we get the required formula. □

An L-minimal surface enveloped by a hyperbolic family of cones is obtained from the surface (4.16) by “transformation” (2.2). Let us give some typical examples obtained from the graphs of the functions in the left column of Table 4; see also Figure 7. These examples are building blocks forming all the surfaces in question.

Example 4 The first example is the catenoid. It can be parametrized as

$$\mathbf{r}_4(u,v) = \left(u + \frac{u}{u^2+v^2},\ v + \frac{v}{u^2+v^2},\ \ln(u^2+v^2) \right). \tag{4.17}$$

Its cyclographic preimage can be written as

$$\mathbf{R}_4(\varphi,\lambda) = (\,0,\ 0,\ -2\varphi,\ -2\,) + \lambda\,(\,0,\ 0,\ \cosh\varphi,\ \sinh\varphi\,). \tag{4.18}$$

Example 5 Another building block is given by the surface $\mathbf{r}_5$ parametrized by:

$$\mathbf{r}_5(u,v) = \left((u^2-v^2)\left(1-\frac{1}{u^2+v^2}\right) - \ln(u^2+v^2),\ 2uv\left(1-\frac{1}{u^2+v^2}\right),\ 4u \right). \tag{4.19}$$

Its cyclographic preimage is the surface parametrized by

$$\mathbf{R}_5(\varphi,\lambda) = \left(\,1 - e^{-2\varphi} + 2\varphi,\ 0,\ 0,\ 0\,\right) + \lambda\,(\,0,\ 0,\ \cosh\varphi,\ \sinh\varphi\,). \tag{4.20}$$

Example 6 Finally we have the surface $\mathbf{r}_6$ given implicitly by $z^2(1-x+z^2/4) = y^2$. In parametric form it can be written as:

$$\mathbf{r}_6(u,v) = \left((u^2-v^2)\left(1-\frac{1}{u^2+v^2}\right)^2,\ 2uv\left(1-\frac{1}{u^2+v^2}\right)^2,\ 4u\left(1-\frac{1}{u^2+v^2}\right) \right). \tag{4.21}$$

Its cyclographic preimage can be written as

$$\mathbf{R}_6(\varphi,\lambda) = (\,2 - 2\cosh 2\varphi,\ 0,\ 0,\ 0\,) + \lambda\,(\,0,\ 0,\ \cosh\varphi,\ \sinh\varphi\,). \tag{4.22}$$

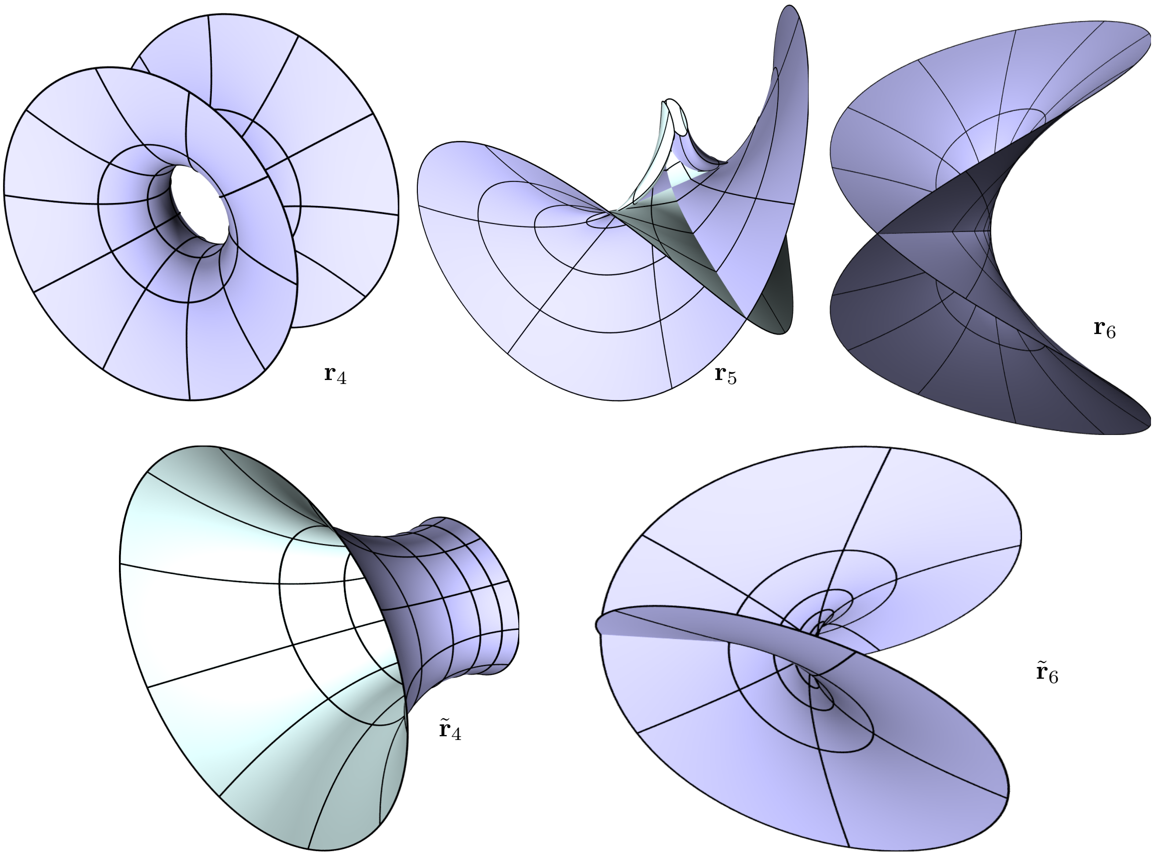

Fig. 7 Building blocks for L-minimal surfaces enveloped by a hyperbolic family of cones. Starting from the top left we show the surfaces $\mathbf{r}_4$, $\mathbf{r}_5$, $\mathbf{r}_6$, $\tilde{\mathbf{r}}_6$ and $\tilde{\mathbf{r}}_4$ in clockwise direction. For details refer to Examples 4, 5, 6 and Section 2.2.

An L-minimal surface enveloped by a hyperbolic family of cones can be obtained from Examples 4–6 by performing L-transformations and taking special convolution surfaces; see Figure 1:

Corollary 5 (Classification for hyperbolic type) *A Laguerre minimal surface enveloped by a hyperbolic family of cones is Laguerre equivalent to a piece of the surface*

$$\mathbf{r}(u,v) = a_1\mathbf{r}_4(u,v) + a_2\mathbf{r}_5(u,v) + a_3\mathbf{r}_6^{\theta}(u,v) + a_4\tilde{\mathbf{r}}_4(u,v) + a_5\tilde{\mathbf{r}}_6^{\vartheta}(u,v) \tag{4.23}$$

for some $a_1, a_2, a_3, a_4, a_5, \theta, \vartheta \in \mathbb{R}$. Conversely, an immersed piece of surface (4.23) is Laguerre minimal and is enveloped by a hyperbolic family of cones.

Proof (of Corollary 5) Let us prove the direct implication. Let Φ be an L-minimal surface enveloped by a hyperbolic family of cones. Then the surface Φ^i carries a family of i-M-circles such that the top view of the family is a hyperbolic pencil. By Theorem 8 it follows that the surface Φ^i is i-M-equivalent to surface (4.16).

The right-hand side of formula (4.16) is a linear combination in the expressions in the left column of Table 4. Performing an i-M-transformation $z \mapsto z + a(x^2+y^2) + bx + cy + d$ one can eliminate the last expression from the linear combination. By Proposition 1 and Table 2 transformation (2.2) takes the functions in the left column of Table 4 to the surfaces in the right column. Since the transformation (2.2) is linear in F the direct implication follows.

The converse implication follows from Propositions 1, 2, Theorem 2, and Table 4. □

There is a simple parametrization of the cyclographic preimage (the proof is analogous to the proof of Theorem 7).

Theorem 9 *The cyclographic preimage of a Laguerre minimal surface enveloped by a hyperbolic family of cones is up to Lorentz transformations a piece of the surface*

$$\mathbf{R}(\varphi,\lambda) = (\,A\varphi + B\cosh 2\varphi,\; C\varphi + D\cosh 2\varphi + E\sinh 2\varphi,\; F\varphi,\; G\varphi\,) + \lambda\,(\,0,\;0,\;\cosh\varphi,\;\sinh\varphi\,) \tag{4.24}$$

for some $A, B, C, D, E, F, G \in \mathbb{R}$.

4.3 Parabolic families of cones

Definition 4 A *parabolic* pencil of circles in the plane (or in a sphere) is the set of all the circles touching a fixed circle at a fixed point. A 1-parametric family of cones (possibly degenerating to cylinders or lines) in space is *parabolic* if the Gaussian spherical images of the cones form a parabolic pencil of circles in the unit sphere.

Theorem 10 *Let* Φ^i *be an i-Willmore surface carrying a family of i-M-circles. Suppose that the top view of the family is a parabolic pencil of circles. Then the surface* Φ^i *is i-M-equivalent to a piece of the surface*

$$z = a_1(5y^2 - x^2)x^3 + a_2(3y^2 - x^2)x^2 + (b_1y^2 + b_2xy + b_3x^2)x + c_1y^2 + c_2xy \tag{4.25}$$

for some $a_1, a_2, b_1, b_2, b_3, c_1, c_2 \in \mathbb{R}$.

Proof (of Theorem 10) Perform an i-M-transformation taking the parabolic pencil of circles in the top view into the pencil of parallel lines $x = t$, where t runs through a segment $J \subset \mathbb{R}$. Denote by $z = F(x, y)$ the surface obtained from the surface Φ^i by the transformation, where F is a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Since an i-M-transformation takes i-M-circles to i-M-circles it follows that the restriction of the function F to (an appropriate segment of) each line $x = t$, where $t \in J$, is a quadratic function.

So $F(x, y) = a(x)y^2 + b(x)y + c(x)$. Thus $\Delta^2 F = a^{(4)}y^2 + b^{(4)}y + c^{(4)} + 4a''$. Since $\Delta^2 F = 0$ it follows that the coefficients of this polynomial in y vanish. Hence

$$\begin{aligned} a(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3;\\ b(x) &= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3;\\ c(x) &= \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 - \alpha_2 x^4/3 - \alpha_3 x^5/5. \end{aligned}$$

One can achieve $\beta_0 = \gamma_0 = \gamma_1 = \gamma_2 = 0$ by the i-M-transformation $z \mapsto z - \gamma_2(x^2 + y^2) - \gamma_1 x - \beta_0 y - \gamma_0$. We get the required formula. □

Table 5 Biharmonic functions whose restrictions to each line $x = t$, $t \in I$, are quadratic functions and corresponding Laguerre minimal surfaces

Biharmonic function	Laguerre minimal surface
$(x\cos\theta + y\sin\theta)^2/2$	$\mathbf{r}_7^{\theta}(u, v)$
x^3	$\mathbf{r}_8(u, v)$
x^2y	$\mathbf{r}_9(u, v)$
xy^2	$\mathbf{r}_9^{\pi/2}(u, v)$
$x^2(x^2 - 3y^2)/2$	$\mathbf{r}_{10}(u, v)$
$x^3(x^2 - 5y^2)$	$\mathbf{r}_{11}(u, v)$
$a(x^2 + y^2) + bx + cy + d$	oriented sphere

An L-minimal surface enveloped by a parabolic family of cones is obtained from the surface (4.25) by transformation (2.2). Let us give some typical examples obtained from the graphs of the functions in the left column of Table 5; see also Figure 8. These examples are building blocks forming all the surfaces in question.

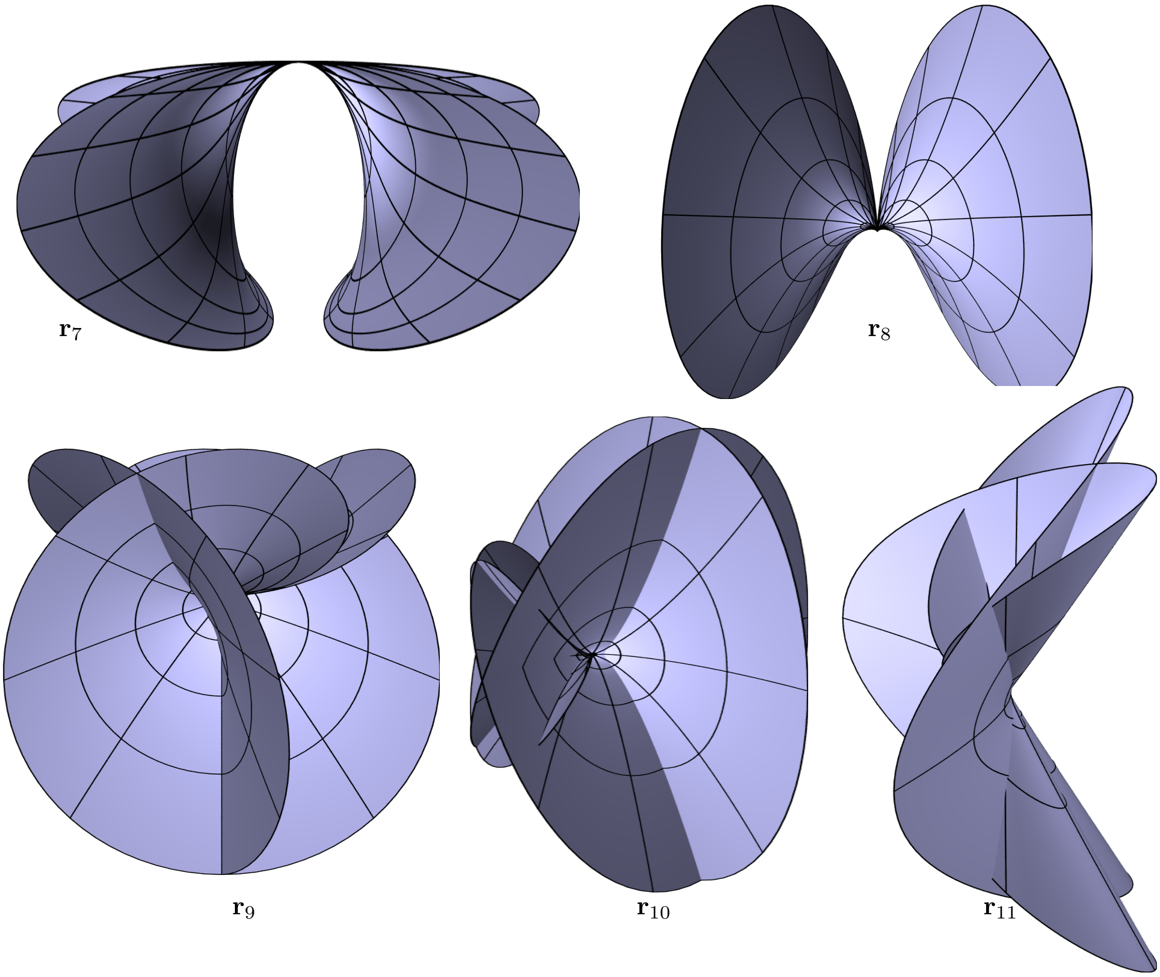

Fig. 8 Building blocks for L-minimal surfaces enveloped by a parabolic family of cones. Starting from the top left we show the surfaces $\mathbf{r}_7$, $\mathbf{r}_8$, $\mathbf{r}_{11}$, $\mathbf{r}_{10}$ and $\mathbf{r}_9$ in clockwise direction. For details refer to Examples 7–11.

Example 7 The first example is the parabolic horn cyclide $\left(y^2+z^2\right)(1-z)=x^2 z$. In parametric form it can be written as:

$$\mathbf{r}_7(u,v)=\frac{1}{1+u^2+v^2}\left(-u-uv^2,\, u^2v,\, u^2\right). \tag{4.26}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_7(\varphi,\lambda)=\left(0,\, 0,\, -\varphi^2,\, \varphi^2\right)/2\,+\,\lambda\,(1,\, 0,\, -\varphi,\, \varphi) \tag{4.27}$$

The surface $\mathbf{r}_7$ has the following property: there is a 2-parametric family of cones touching the cyclide along certain curves. In particular, there are both parabolic and elliptic 1-parametric families of cones touching the surface along curves. One of the elliptic families of cones can be parametrized as

$$\left(0,\, 0,\, \cos^2\varphi,\, \cos^2\varphi\right)/2\,+\,\lambda\,(\sin\varphi,\, \cos\varphi,\, 0,\, 0) \tag{4.28}$$

Example 8 The next building block is given by the algebraic surface of degree 6 with implicit equation

$$(x^2+y^2+z^2)z^4-2(8x^2+9y^2+9z^2)xz^2-27(y^2+z^2)^2=0.$$

In parametric form it can be written as:

$$\mathbf{r}_8(u,v)=\frac{1}{1+u^2+v^2}\left(u^4-3u^2v^2-3u^2,\, 4u^3v,\, 4u^3\right). \tag{4.29}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_8(\varphi,\lambda) = \big(\,0,\;0,\;-\varphi^3,\;\varphi^3\,\big) + \lambda\,(1,\;0,\;-\varphi,\;\varphi\,) \tag{4.30}$$

Example 9 Another example is the algebraic surface of degree 8, given implicitly by:

$$z^2(y^2+z^2)\left(z^2-4y-4\right)^2 + x^2\left(64y^3-24(y-3)yz^2-6(y+6)z^4+z^6\right) - 27x^4z^2 = 0.$$

In parametric form:

$$\mathbf{r}_9(u,v) = \frac{1}{1+u^2+v^2}\,\big(\,2uv(u^2-v^2-1),\;u^2(3v^2-u^2-1),\;4u^2v\,\big)\,. \tag{4.31}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_9(\varphi,\lambda) = \big(\,0,\;-\varphi^2,\;0,\;0\,\big) + \lambda\,(1,\;0,\;-\varphi,\;\varphi\,) \tag{4.32}$$

This surface has the following property: there are two 1-parametric families of cones touching the surface along certain curves. The other family can be parametrized as

$$\big(\,0,\;0,\;\varphi+\varphi^3,\;\varphi-\varphi^3\,\big) + \lambda\,(0,\;1,\;-\varphi,\;\varphi\,)\,. \tag{4.33}$$

Finally, we have the following two "monsters". We do not write their implicit equations because this would take several pages.

Example 10 First the algebraic surface of degree not greater than 14 described by

$$\mathbf{r}_{10}(u,v) = \frac{1}{1+u^2+v^2}\begin{pmatrix} u^5-2u^3(1+4v^2)+3u(v^2+v^4) \\ 3u^2v(1+2u^2-2v^2) \\ 3u^2(u^2-3v^2) \end{pmatrix}. \tag{4.34}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_{10}(\varphi,\lambda) = \big(\,0,\;0,\;-3\varphi^2-4\varphi^4,\;-3\varphi^2+4\varphi^4\,\big)/2\;+\;\lambda\,(1,\;0,\;-\varphi,\;\varphi\,)\,. \tag{4.35}$$

Example 11 The second monster is the algebraic surface of degree not greater than 18 with parametrization

$$\mathbf{r}_{11}(u,v) = \frac{1}{1+u^2+v^2}\begin{pmatrix} 3u^6-5u^4(1+6v^2)+15u^2(v^2+v^4) \\ 2u^3v(5+9u^2-15v^2) \\ 8u^3(u^2-5v^2) \end{pmatrix}. \tag{4.36}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_{11}(\varphi,\lambda) = \big(\,0,\;0,\;-5\varphi^3-6\varphi^5,\;-5\varphi^3+6\varphi^5\,\big) + \lambda\,(1,\;0,\;-\varphi,\;\varphi\,)\,. \tag{4.37}$$

An L-minimal surface enveloped by a parabolic family of cones can be obtained from Examples 7–11 by performing rotations and taking special convolution surfaces; see Figure 9:

Corollary 6 (Classification for parabolic type) *A Laguerre minimal surface enveloped by a parabolic family of cones is Laguerre equivalent to a piece of the surface*

$$\mathbf{r}(u,v) = a_1\mathbf{r}_7^{\theta}(u,v) + a_2\mathbf{r}_8(u,v) + a_3\mathbf{r}_9(u,v) + a_4\mathbf{r}_9^{\pi/2}(u,v) + a_5\mathbf{r}_{10}(u,v) + a_6\mathbf{r}_{11}(u,v) \tag{4.38}$$

for some $a_1, a_2, a_3, a_4, a_5, a_6, \theta \in \mathbb{R}$. *Conversely, an immersed piece of surface (4.38) is Laguerre minimal and is enveloped by a parabolic family of cones.*

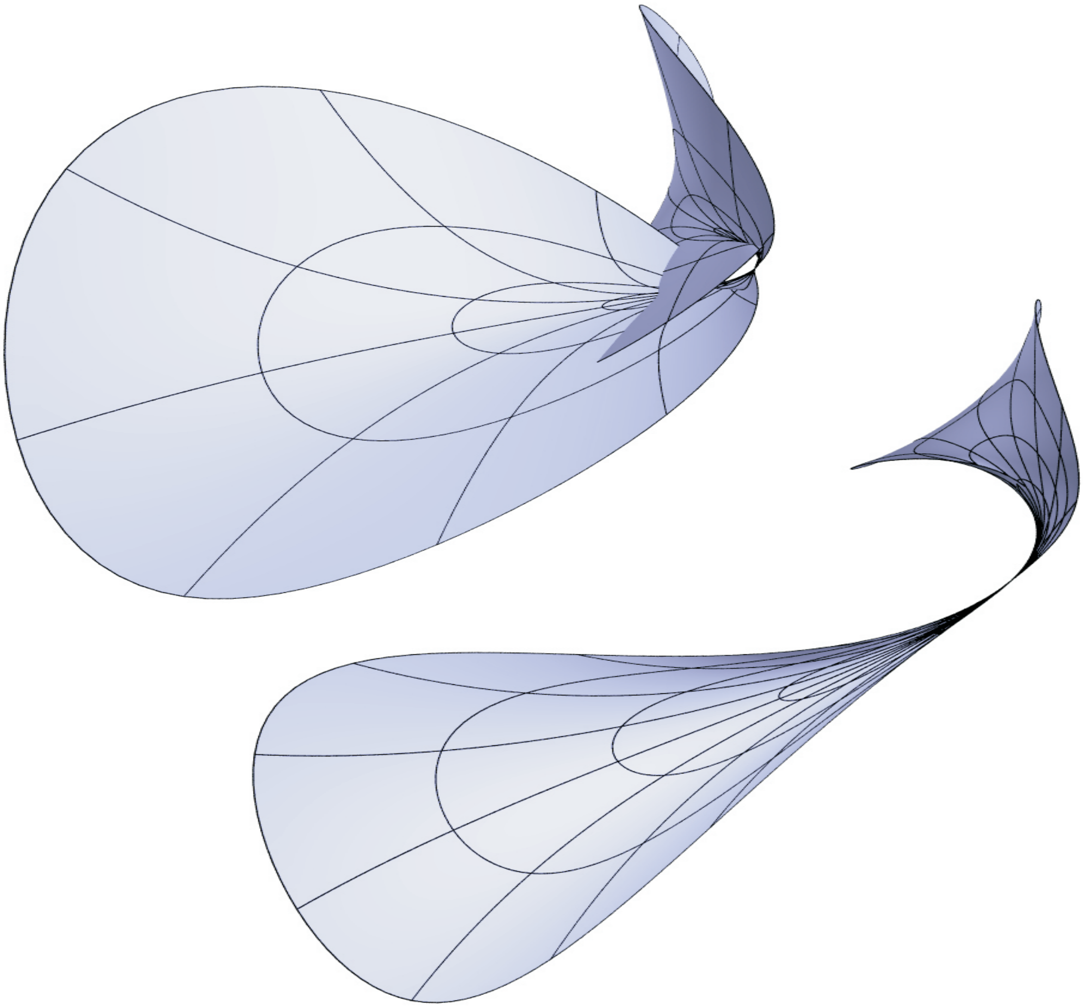

Fig. 9 Two general L-minimal surfaces enveloped by parabolic families of cones. For details refer to Definition 4 and Corollary 6.

Proof (of Corollary 6) Let us prove the direct implication. Let Φ be an L-minimal surface enveloped by a parabolic family of cones. Then the surface Φ^i carries a family of i-M-circles such that the top view of the family is a parabolic pencil. By Theorem 10 it follows that the surface Φ^i is i-M-equivalent to surface (4.25).

Right-hand side of formula (4.25) is a linear combination in the expressions in the left column of Table 5. Performing an i-M-transformation $z \mapsto z + a(x^2 + y^2) + bx + cy + d$ one can eliminate the last expression from the linear combination. By Propositions 1 and Table 2 transformation (2.2) takes the functions in the left column of Table 5 to the surfaces in the right column. Since the transformation (2.2) is linear in F the direct implication follows.

The converse implication follows from Propositions 1, 2, Theorem 2, and Table 5. $\square$

Finally we describe the cyclographic preimage (the proof is analogous to the proof of Theorem 7).

Theorem 11 *The cyclographic preimage of a Laguerre minimal surface enveloped by a parabolic family of cones is up to Lorentz transformations a piece of the surface*

$$\mathbf{R}(\varphi, \lambda) = \begin{pmatrix} 0 \\ A\varphi + B\varphi^2 \\ C\varphi + D\varphi^2 + E\varphi^3 + F(3\varphi^2 + 4\varphi^4) + G(5\varphi^3 + 6\varphi^5) \\ C\varphi - D\varphi^2 - E\varphi^3 + F(3\varphi^2 - 4\varphi^4) + G(5\varphi^3 - 6\varphi^5) \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -\varphi \\ \varphi \end{pmatrix} \tag{4.39}$$

for some $A, B, C, D, E, F, G \in \mathbb{R}$.

4.4 Open problems

Conjecture 1 A surface such that there is a 2-parametric family of cones of revolution touching the surface along certain curves distinct from directrices is either a sphere or a parabolic cyclide.

Problem 1 Describe all surfaces such that there are two 1-parametric families of cones of revolution touching the surface along curves.

Problem 2 Describe all Willmore surfaces such that there is a 1-parametric family of circles lying in the surface.

Acknowledgements

The authors are grateful to S. Ivanov for useful discussions. M. Skopenkov was supported in part by Möbius Contest Foundation for Young Scientists and the Euler Foundation. H. Pottmann and P. Grohs are partly supported by the Austrian Science Fund (FWF) under grant S92.

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